THE LARGE SCALE GEOMETRY OF THE HIGHER BAUMSLAG-SOLITAR GROUPS

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Introduction

The Baumslag-Solitar groups:

$$BS(m,n) = < x, y | xy^m x^{-1} = y^n >$$

are some of the simplest interesting infinite groups which are not lattices in Lie groups. They have been studied in depth from the point of view of combinatorial group theory. It is natural to ask if the geometric approach to the theory of infinite groups, which has been so successful in the study of lattices, can yield any insights in this nonlinear case.

The first step towards a geometric understanding of the Baumslag-Solitar groups is to decide which among the $BS(m,n)$ are quasi-isometric. The groups $BS(1,n)$ are solvable, hence amenable, and so are not quasi-isometric to any of the $BS(m,n)$ with $1 < m \leq n$ which contain free subgroups and hence are are nonamenable.

The solvable groups $BS(1,n)$ are in many respects the most lattice-like of the Baumslag-Solitar groups. They are discrete subgroups in products of real and p-adic Lie groups. The groups $BS(1,n)$ are classified up to quasi-isometry by Farb and Mosher in [FM1]. They prove that $BS(1,n)$ and $BS(1,m)$ are quasi-isometric only if n and m have common powers. When n and m have common powers $BS(1,n)$ and $BS(1,m)$ are not only quasi-isometric, but are commensurable (have isomorphic subgroups of finite index). This is the same rigidity phenomenon as occurs for nonuniform lattices in higher rank. Despite this rigidity, their full group of self quasi-isometries is quite large, and in this they more closely resemble uniform lattices.

In this paper we classify all the Baumslag-Solitar groups up to quasi-isometry. The higher Baumslag-Solitar groups, namely those with $1 < m < n$, are unlike the groups $BS(1,n)$ in many ways. They are nonlinear, not residually finite, and usually not Hopfian. Indeed, this "bad" behavior was the motivation for their discovery. Our results show that the higher Baumslag-Solitar groups exhibit a surprising lack of rigidity; all the higher Baumslag-Solitar groups, aside from the degenerate case of $BS(n,n)$, are quasi-isometric to each other. The quasi-isometries we construct do not reflect any clear algebraic relationship between the groups. In particular, many of the groups we prove to be quasi-isometric are not commensurable.

Our method of constructing quasi-isometries seems to be fundamentally different from earlier constructions.

Our main results concern a class of groups somewhat larger than the class of Baumslag-Solitar groups. We define a *graph of* $\mathbb{Z}s$ as a finite graph of groups, in the sense of Serre ([Se]), with all vertex and edge groups infinite cyclic. This class includes the Baumslag-Solitar groups, which are precisely the HNN extensions of $\mathbb{Z}$.

Theorem 0.1 (Classification of Graphs of $\mathbb{Z}$s). *If G is a graph of $\mathbb{Z}s$ and $\Gamma = \pi_1 G$ then exactly one of the following is true:*

(1) *Γ contains a subgroup of finite index of the form $F_n \times \mathbb{Z}$.*
(2) *$\Gamma = BS(1,n)$ for some $n > 1$.*
(3) *Γ is quasi-isometric to $BS(2,3)$.*

Here F_n is the free group on n generators.

Corollary 0.2 (Classification of Higher Baumslag-Solitar Groups). *All the groups $BS(m,n)$ with $1 < m < n$ are quasi-isometric to each other.*

0.1. **Outline.** Let G be the fundamental groups of a graph of $\mathbb{Z}$s. We begin by constructing a geometric model space for the geometry of G. This model is a contractible 2-complex, X_G on which G acts cocompactly, freely and properly discontinuously by isometries. The algebraic fact that G is a graph of $\mathbb{Z}$s translates into the geometric fact that X_G is a warped product of a tree with $\mathbb{R}$. In other words, X_G is topologically $T \times \mathbb{R}$, with a metric which differs from the product metric in that the metric on $v \times \mathbb{R}$ is scaled by a warping function $T \to \mathbb{R}^+$.

The tree T is the Bass-Serre tree of the graph of groups, and the warping function is induced by a G invariant orientation on T. If two graphs of $\mathbb{Z}$s are quasi-isometric, we show that there is a quasi-isometry between their Bass-Serre trees which coarse respects, in an appropriate sense, the orientations. Conversely, any coarsely orientation preserving quasi-isometry between Bass-Serre trees induces a quasi-isometry between the groups. Thus the classification of Baumslag-Solitar groups, and graphs of $\mathbb{Z}$s in general, reduces to classifying coarsely oriented trees.

The heart of our construction is the construction of coarsely orientation preserving quasi-isometries between trees. We first decompose the trees into lines of constant "slope" (see §2.1) with respect to the orientation. The quasi-isometries are built line by line. This also requires a quasi-isometry between the spaces of lines, with nice properties with respect to the orientation. Building this matching of lines uses Hall's Selection theorem, and the axiom of choice.

This construction is sufficient to allow us to completely classify graphs of $\mathbb{Z}$s up to quasi-isometry. We also explore the issue of commensurabilities among graphs of $\mathbb{Z}$s sufficiently to show that although all the groups

$BS(m,n)$ for $1 < m < n$ are quasi-isometric, they are in general not commensurable. Thus we have many new explicit examples of groups which are quasi-isometric but not commensurable.

We next turn to describing the quasi-isometry groups of these graphs of $\mathbb{Z}$s. For the quasi-isometry groups of the solvable Baumslag-Solitar groups, $BS(1,n)$, there is a nice description in [FM1]. The situation for the higher Baumslag-Solitar groups is substantially more complex. We discuss the complications and give several descriptions, none entirely satisfactory, of these quasi-isometry groups.

We discuss some generalizations. It is natural to ask what an arbitrary finitely generated groups quasi-isometric to a graph of $\mathbb{Z}$s can be. It follows from [MSW] that any such group is a finite graph of virtual $\mathbb{Z}$s. As our classification extends without change to that larger class of groups, we get a complete description of the class of groups quasi-isometric to the higher Baumslag-Solitar groups. We also discuss some further classes of graphs of groups to which our classification of coarsely oriented trees is relevant.

1. The geometric models

One of the basic principles of geometric group theory is the Milnor-Svarc theorem, which says that if G is a finitely generated group, and G acts properly discontinuously and cocompactly by isometries on a proper geodesic metric space X, then X is quasi-isometric to G. Thus, for questions about the quasi-isometric geometry of G, one can work instead with X.

1.1. **The 2-complexes.**

Let Γ be a graph of $\mathbb{Z}$s, $G = \pi_1\Gamma$, and T the Bass-Serre tree of G.

We first describe a 2-complex X_G on which G acts properly discontinuously and cocompactly by isometries. Build a compact complex with $\pi_1 = G$ out of the graph Γ as follows: start with a disjoint collection of circles, one for each vertex of Γ. For each edge of Γ glue in an $S^1 \times [0,1]$ where the attaching maps at each end are covering maps inducing the same map on fundamental groups as the inclusions of the corresponding edge groups. The universal cover of the complex is the desired X_G.

Following [FM1], we give another description of X_G. Topologically, X_G is $T \times \mathbb{R}$. Let e be an edge of Γ, and let the index of the inclusion into its vertex groups be $n \geq m$. The action of the edge group of e on the strip $e \times \mathbb{R}$ is translation by n over one endpoint and by m over the other. This becomes isometric if we metrize the strip as a warped product $dt^2 + (\frac{n}{m})^{2t} ds^2$, where t is the parameter along e, and x along $\mathbb{R}$. This makes $e \times \mathbb{R}$ isometric to a horostrip (the region between two concentric horoballs) of width 1 in a space of constant curvature $-\ln \frac{n}{m}$.

For any vertex v in T, we call the subspace $v \times \mathbb{R}$ of X_G the *vertex space* over v. Likewise, for any edge e, $e \times \mathbb{R}$ is the *edge space* over e.

Given any two vertices of T, v_1 and v_2, let G_1 and G_2 be their stabilizers. Let G_{12} be their intersection, which is the stabilizer of the path between them. G_{12} has finite intersection in both G_1 and G_2. We call the ratio $\frac{[G_{12}:G_1]}{[G_{12}:G_2]}$ the *contraction factor* between v_1 and v_2. The terminology is justified by the geometric interpretation as the contraction factor of the closest point projection map from the vertex space over v_1 to the vertex space over v_2.

It is more convenient to work with an additive rather than multiplicative invariant. We define the *height change* between two vertices as the logarithm of the contraction factor. By choosing a base point of T, we can define the *height* of a vertex v as the height change between the base point and v. We extend the height function, h, to all of T by linear interpolation along edges.

The metric on X_G can be described in terms of the height function as a warped product $T \propto \mathbb{R}$ with warping function e^{-h}. Thus T, together with the height function, determines the complex X_G up to isometry.

1.2. **The coarsely oriented Bass-Seree tree.** We view the height change along edges as giving a quantitative analogue of an orientation. If S is an oriented tree, then we can define a height change function by declaring the height change across an edge to be 1 or -1 depending on the orientation. With this height change function, an isometry of S preserves the orientation if and only if it preserves the height change function. Just as we view a quasi-isometry as a "large scale isometry", we view a quasi-isometry which preserves the height change function, on a large scale, as coarsely orientation preserving.

Definition A quasi-isometry $f : T_1 \to T_2$ between trees with height functions h_1 and h_2, is *coarsely orientation preserving* is there is $C > 0$ so that for all v_1 and v_2 in T_1:

$$|h_1(v_1, v_2) - h_2(f(v_1), f(v_2))| \leq C$$

Notice that only the height change between two points is involved in this definition, so the notion of coarsely orientation preserving is independent of the choice of base points.

Theorem 1.1. *For $i = 1, 2$, let G_i be a graph of $\mathbb{Z}$s, with Bass-Serre tree T_i. If f is a quasi-isometry from T_1 to T_2 which is coarsely orientation preserving then $f \times Id : X_{G_1} \to X_{G_2}$ is a quasi-isometry.*

Proof. The cases of bounded and unbounded height functions are fundamentally different. In the former, the vertex spaces are isometrically embedded, while in the latter they are exponentially distorted. Clearly, boundedness of height function is a coarse orientation preserving quasi-isometry invariant. The theorem holds in this case as, when the height function is bounded, the complex X is bilipschitz equivalent to the product $T \times \mathbb{R}$. Thus we assume the height functions are both unbounded.

Lemma 1.2 (Metric Approximation). *If the height function on T is unbounded then the distance in X_G is quasi-isometric to:*

$$d_T(t_1,t_2) + max(0, -h(t_1,t_2) + \ln|x_1 - x_2|)$$

where $h(t_1,t_2)$ is the maximum height along the geodesic t_1t_2.

Proof. As the set of vertex spaces is coarsely dense in X, we may assume that t_1 and t_2 are vertices of T. Given any path p from (t_1,x_1) to (t_2,x_2), we can replace p by a path which is piecewise horizontal (constant $\mathbb{R}$ coordinate) or vertical (constant T coordinate) without multiplying the height by more that a constant factor.

The total length of such a path is the length of its projection to T (= the length of the horizontal segments) plus the length of the lengths of the vertical segments. The length of a vertical segment in the vertex space over t is $e^{-h(t)}$ times the change in the $\mathbb{R}$ coordinate. Thus any path can be shortened by moving all the vertical changes to occur in the vertex space over the point of maximal height on the projection of the path to T. Thus the distance between the points (t_1,x_1) and (t_2,x_2) is bounded below by a multiple of the minimal length of a path which is a horizontal path from t_1 to a point t, followed by a vertical path from (t,x_1) to (t,x_2) an then a horizontal path from t to t_2. The length of such a path is $d_T(t_1,t) + d_T(t_2,t) + e^{-h(t)}|x_1 - x_2|$.

This length is equal to $d_T(t_1,t_2) + 2d_T(t, t_1t_2) + e^{-h(t)}|x_1 - x_2|$. Replacing t by the closest point at the same height as t to t_1t_2 shortens the path, so we may assume that t is this closet point.

Since there is a cocompact symmetry group, it is easy to see that there are $\beta > 0$ and $C > 0$ so that the distance of any point t of T to the set of points of height at least h in T is within C of $\beta max(0, h - h(t))$. Thus the minimal length of a path from (t_1,x_2) to (t_2,x_2) is, to within C, the minimum over h of:

$$d_T(t_1,t_2) + 2max(0, \beta|h - h(t_1,t_2)|) + e^{-h}|x_1 - x_2|$$

The minimum of this over all h occurs at $h = h(t_1,t_2)$ if $|x_1 - x_2| \leq \frac{2e^{h(t_1,t_2)}}{\beta}$, and at $h = \ln\frac{\beta|x_1-x_2|}{2}$ otherwise. Substituting this value for h finishes the proof of the lemma.

□

Using this lemma, we complete the proof of the theorem. By choosing basepoints so that f is basepoint preserving, we may assume the difference $|h_1(t) - h_2(f(t))|$ is bounded. Since f is a quasi-isometry, the image of the geodesic from t to t' is within a uniformly bounded distance of the geodesic from $f(t)$ to $f(t')$. Combining these facts, we see that the difference between $h_1(t,t')$ and $h_2(f(t),f(t'))$ is uniformly bounded. The approximation to the distance in X in the lemma is thus quasi-preserved by $f \times Id$, and thus $f \times Id$ is a quasi-isometry.

□

Thus, if we can construct a coarse orientation preserving quasi-isometry between the Bass-Serre trees of two graphs of $\mathbb{Z}$s, this gives a quasi-isometry between the groups. In fact, all quasi-isometries among graphs of $\mathbb{Z}$s arise this way, see §4.

2. Constructing quasi-isometries

Definition A coarsely oriented tree is *homogeneous* if the multiset of height changes of edges incident to a vertex v is the same for all v. This is equivalent to the transitivity of height change preserving isometries.

The Bass-Serre trees of the Baumslag-Solitar groups are homogeneous. We show in section §3.1 that any coarsely oriented tree with cocompact symmetry group is coarsely orientation preserving quasi-isometric to a homogeneous tree.

In this section we classify homogeneous coarsely oriented tree up to coarsely orientation preserving quasi-isometry. Recall that is T is an oriented tree, there is an induced coarse orientation in which the height change across an edge is either 1 or -1 depending on whether is edge is crossed with or against the orientation. Homogeneous oriented trees are determined by their *type*, which is the ordered pair (n, m) of the number of edges oriented away from and the number of edges oriented towards any vertex.

Theorem 2.1 (Classification of Homogeneous Trees). *Let T be a homogeneous coarsely oriented tree with height function h. Precisely one of the following holds:*

- *h is constant.*
- *At every vertex of T there is one edge which strictly increases (resp. decreases) height, and all the other edges at the vertex strictly decrease (resp. increase) height.*
- *T is coarsely orientation preserving quasi-isometric to the oriented tree of type $(2, 2)$.*

Proof. The bulk of the proof of this theorem is constructing coarsely orientation preserving quasi-isometries to show

Lemma 2.2 (The Main Lemma). *If T is a homogeneous coarsely oriented tree for which, at every vertex, there are at least two edges which strictly increase height, and two edges which strictly decrease height, then T is coarsely orientation preserving quasi-isometric to the homogeneous oriented tree of type $(2, 2)$.*

Assuming the lemma we complete the proof of the theorem.

If there are no edges which change height, then h is constant. Otherwise there are, at every vertex, both an edge which strictly increases height and an edge which strictly decreases height.

Suppose there are edges which do not change height. Let F be the forest of such edges. The components of F are either edges, with one at every vertex, or infinite trees without valence one vertices. In the former case, collapsing F is a coarse orientation preserving quasi-isometry to a tree which satisfies the hypothesis of lemma 2.2. In the latter case, the following lemma produces a subset F, the collapsing of which has the same result.

Lemma 2.3. *Let S be an infinite tree without valence one vertices. There is a subset of the edges of S which contains exactly one edge at every vertex.*

Proof. Let S' be a maximal subtree of S for which there is such a subset of edges. If $S' \neq S$ then there is a vertex v of $S - S'$ and an edge e with one endpoint v and the other endpoint, u, in S'. If u is not in an edge of the subset, then one can extend S' to $S' \cup e$ and add e to the subset of edges. If u is in one of the subset of edges of S' then let e' be any edge at v other than e, and extend S' to $S' \cup e \cup e'$ adding e' to the subset of edges. In either case this contradicts maximality. □

Finally, if there are no edges which do not change height, then one is clearly either in the second case of the theorem or satisfy the hypotheses of lemma 2.2 and hence in the third case. This completes the proof of the theorem, assuming lemma 2.2.

□

Proof. We now turn to the proof of lemma 2.2. There are two steps in this proof. The first step is to decompose the tree into lines along which the height function changes at essentially a constant rate with respect to length. The second step is to find a matching of the lines in one tree with the lines in the other so that we can assemble a coarsely orientation preserving quasi-isometry line by line.

2.1. **Constant slope laminations.** bf Definition Let β and C in $\mathbb{R}$ be given. We call a bi-infinite geodesic γ in T a *line of slope* (β, C) if and only if for all n and m in $\mathbb{Z}$

$$|h(\gamma(n)) - h(\gamma(m)) - \beta(n-m)| \leq C$$

Theorem 2.4 (Existence of constant slope laminations)**.** *If T is a homogeneous tree with height function which has at each vertex at least two edges along which the height increases, and two along which it decreases, then there is $\beta_0 > 0$ so that for any $0 \leq \beta \leq \beta_0$ there is a C and a family of lines of slope (β, C) exactly one of which passes through each vertex of T.*

We will call such a collection a *lamination by lines of slope β*.

Proof. Take β_0 to be such that there are two or more edges, at each vertex, which increase height by at least β_0 and two or more which decrease it by at least β_0. Fix $0 \leq \beta \leq \beta_0$. Let M be the maximal amount height changes along any edge, and take $C = 2M$.

Given any vertex v and edge e at v which increases height by at least β_0 we can find a ray of slope (β, M) starting at v and beginning with e. We build this ray inductively. If a ray r of length n has been constructed, extend it to length $n+1$ by choosing an edge which increases height by at least β if $\beta(n+1) \geq h(r(n)) - h(v)$ and choosing one which decreases height by at least β otherwise. It is easy to see that r has the desired properties. It is likewise possible to build a ray of slope $(-\beta, M)$ through any edge e' at v along which height decreases by at least β. By gluing the two we get a line of slope (β, C).

Now suppose we have T' a subtree of T which has been given a covering by lines of slope (β, C). If $T' \neq T$ then there is a v, a vertex of T, which is adjacent to T'. Since only one edge connects v to T' we can build a line of slope (β, C) through v disjoint from T'. Then we can enlarge T' to include v, the edge connecting v to T', and the new line. Continuing in this way we cover all of T. □

2.2. **Matching the lines.** Given two trees, T_1 and T_2, covered by lines of slope β_1 and β_2 we try to find an coarsely orientation preserving quasi-isometry from T_1 to T_2 one line at a time. Given two lines there is an coarsely orientation preserving quasi-isometry between the lines, which is unique up to bounded distance. Given a bijection between the sets of lines covering T_1 and those covering T_2 we get almost orientation preserving maps $T_1 \to T_2$ and $T_2 \to T_1$ with compositions at bounded distance from the identity maps of T_1 and T_2. We now discuss the precise conditions which make this map a quasi-isometry of the trees.

Let T_1' and T_2' be the trees obtained from T_1 and T_2 by collapsing the lines of the laminations to points. Suppose we have a tree isomorphism, f, between these quotients. This gives, as above, $\hat{f} : T_1 \to T_2$. This $\hat{f}$ has bounded stretch along the lines of the laminations and is coarsely orientation preserving. If we have an edge e at height h in T_1 which connects two lines, a and b, it maps to an edge of T_1' and so its image under $\hat{f}$ maps to an edge of T_2'. There is a unique edge e' of T_2 which maps to edge of T_2'. The edge e' connects two lines, a' and b', in T_2.

Since $\hat{f}$ is coarsely orientation preserving, the end points of e' are near the points of height h on a' and b'. If e' is at height h' then these points are at distance $2|h' - h| + 1$ in T_2. Thus $\hat{f}$ is a quasi-isometry of T_1 and T_2 if, for every edge e of T_1', the heights of the edge in T_1 mapping to e and the edge of T_2 mapping to $f(e)$ differ by a uniformly bounded amount.

Proposition 2.5. *For $i = 1, 2$ let T_i be a homogeneous tree of valence n_i and be covered by lines of slopes β_i. If $\frac{\beta_1}{\beta_2} = \frac{n_1-2}{n_2-2}$ then there are a $K > 0$ and a tree isomorphism between T_1' and T_2' so that corresponding edges, when lifted to T_1 and T_2, differ in height by at most K.*

Proof. Pick base points v_1 in T_1' and in T_2', let $f(v_1) = v_2$. Assume we can biject the edges at v_1 and v_2 in such a way as to change heights by at most K. This gives f on the balls of radius 1 around the basepoints. Suppose we have the map f defined between the balls of radius n. If, for each v in the sphere of radius n, we can biject the edges at v which connect to the sphere of radius $n+1$ with those at $f(v)$ which connect to the sphere or radius $n+1$, in such a way as to change height by at most K, then we can extend f to the balls of radius $n+1$. Then, by induction, we would have the desired f.

To construct the edge bijections needed in this construction we use Hall's selection theorem. In this context this says that bijections will exist between the edges at w_1 and w_2 if and only if for every interval $[a, b]$ in $\mathbb{R}$ the number of edges at w_1 with heights in $[a, b]$ is no more than the number at w_2 with heights in $[a-K, b+K]$, and vice versa. This holds because the number of vertices on a line l of slope β with heights in the range $[a, b]$ is, to within a uniform additive error, $\frac{b-a}{\beta}$. Thus, by the condition on the slopes, Hall's theorem applies for K large enough. ☐

Lemma 2.4 gives laminations of T_1 and T_2 by lines of constant slope, with slopes of arbitrary ratio. This completes the proof of lemma 2.2. ☐

3. The classification of graphs of $\mathbb{Z}$s

Using theorem 1.1 and the previous construction, we can construct quasi-isometries between many graphs of $\mathbb{Z}$s. For the solvable Baumslag-Solitar groups, the classification in [FM1] proves that quasi-isometry implies abstract commensurability. The quasi-isometries we construct are very different in nature, relying on the axiom of choice. We investigate when these graphs of $\mathbb{Z}$s are commensurable in sufficient detail to see that many of the groups we prove are quasi-isometric are not commensurable. In particular, while all of the higher Baumslag-Solitar groups $BS(m, n)$ for $1 < m < n$ are quasi-isometric, they are, in general, not commensurable.

3.1. **The quasi-isometric classification.**

Proof. Theorem 2.1 allows us to construct the quasi-isometries we need to prove Theorem 0.1. We first show that if G is any graph of $\mathbb{Z}$s then its Bass-Serre tree is coarsely orientation preserving quasi-isometric to a homogeneous tree.

We can assume that there are no edges in the graph of groups, Γ, which have distinct endpoints and for which the edge group includes isomorphically to either of its vertex group. If there were any such edges, they could be collapsed to give a graph of groups with the same fundamental group and fewer edges.

Lemma 3.1. *Let F be a maximal tree in Γ. There is a family of lifts of F to T so that every vertex of T is contained in exactly one of the lifts in the family.*

Proof. This is done exactly as in Theorem 2.4. Since every edge group of F includes as a subgroup of index at least two in both of its vertex groups, for any lift of an endpoint to T there are at least two lifts of the edge at that vertex. If we have lifts which cover a subtree T' of T then there is a v in T

adjacent to T'. As each edge of F has more than one lift at each lift of its endpoints there is a lift of F through v disjoint from T'.

$\square$

Pick a base point in Γ and define the height of a lift of F as the height of the lift of the base point it contains. Then the tree $\hat{T}$ of these lifts, or equivalently the tree obtained by collapsing each lift, is a homogeneous tree coarsely orientation preserving quasi-isometric to T.

If T has bounded height function then it easy to see that G has a subgroup of finite index which is $F_n \times \mathbb{Z}$.

If the height function on T is unbounded then the height function on $\hat{T}$ is also unbounded. Each vertex then must have at least one edge increasing height and one decreasing height. If F contains any edges then there is at least one edge which does not change height at each vertex of the collapsed tree. As in §2 this implies that T is coarsely orientation preserving quasi-isometric to the oriented tree of type $(2, 2)$.

If F contains no edges, then Γ has only one vertex. If there is a loop in Γ which does not change height then again T is coarsely orientation preserving quasi-isometric to the oriented tree of type $(2, 2)$. The same holds, by lemma 2.2, if there are two or more loops that do change height, or a single loop which changes height which has more than one lift at both of its endpoints. Thus the only graph of $\mathbb{Z}$s with unbounded height function not coarsely orientation preserving quasi-isometric to the oriented tree of type $(2, 2)$ is a graph of $\mathbb{Z}$s with a single vertex and a single edge which includes isomorphically at one end. These are precisely the solvable Baumslag-Solitar group, which are classified up to quasi-isometry in [FM1].

This completes the proof of Theorem 0.1. $\square$

3.2. **Noncommensurability.** We investigate when graphs of $\mathbb{Z}$s are commensurable. While we do not get a complete classification, we show that many of the groups we have shown to be quasi-isometric are not commensurable.

Proposition 3.2. *Suppose* $(a, b) = (c, d) = 1$ *and* $\frac{a}{b} \neq \frac{c}{d}$, *then the groups* $BS(a, b)$ *and* $BS(c, d)$ *are not commensurable.*

Let Γ be any graph of $\mathbb{Z}$s not quasi-isometric to $F_n \times \mathbb{Z}$ or to a solvable Baumslag-Solitar group.

An element γ is of *vertex type* if and only if for any $\sigma \in \Gamma$ there are n and m nonzero for which $\gamma^n = \sigma\gamma^m\sigma^{-1}$.

Lemma 3.3. γ *is of vertex type if and only if* γ *stabilizes a vertex.*

Proof. Since the tree has bounded valence any two vertex stabilizers are commensurable, so certainly any element which fixes a vertex is of vertex type. Conversely, if γ does not stabilize a vertex then it is a hyperbolic tree automorphism, so any element which conjugates one power of γ to another must preserve its axis. This can only be the entire group if T is quasi-isometric to $\mathbb{Z}$ which does not happen for graphs of $\mathbb{Z}$s in this quasi-isometry class. $\square$

If Γ is represented by a graph without any edge groups which include isomorphically into either vertex group, for example $BS(m,n)$ for m and n both greater than one, then no vertex stabilizer is contained in another. In that case, the maximal cyclic subgroups of vertex type are precisely the vertex stabilizers, so the vertex set of T is determined as a Γ set by Γ. The height function is also determined, as it is defined in terms of the modular homomorphism which is the ratio of indices of the intersections of two vertex stabilizers in each one. It is not difficult to modify this to cope with loops which include isomorphically into one end. There is some ambiguity in identifying the edges do to the possibility of sliding.

For the special case of the groups $BS(m,n)$ with m and n relatively prime and larger than one, the only finite index subgroups are graphs of $\mathbb{Z}$s with underlying graph a circle and all edge groups including as subgroups of index m and n in its vertex groups. As discussed above, we can therefore recover the number $\frac{n}{m}$ just from the isomorphism type of such a group. Thus we see that $\frac{m}{n}$, and therefore m and n, are commensurability invariants. In other words, no two of these Baumslag-Solitar groups are commensurable.

4. The group quasi-isometries

In this section we calculate the quasi-isometry group of the groups $BS(m,n)$, for $1 < m < n$. As all these groups, and most graphs of $\mathbb{Z}$s, are quasi-isometric they all have the same quasi-isometry group. The quasi-isometry groups of the solvable Baumslag-Solitar $BS(1,n)$ is the product $Bilip(\mathbb{R}) \times Bilip(\mathbb{Q}_n)$ [FM1]. We give a similar description of the quasi-isometry group of the higher Baumslag-Solitar groups, although the final form is substantially more complicated.

We start by proving that the special form of the quasi-isometries we construct in §2 are, in fact, the general case. According to [FM3], if $F : X \to X$ is a quasi-isometry, there is a quasi-isometry $f : T \to T$ so that $\pi(F(x)) = f(\pi(x))$ for π the projection of X to T.

Lemma 4.1. *If $F : X \to X$ is a quasi-isometry covering $f : T \to T$ then f is coarsely orientation preserving.*

Proof. For any t in T, we define the fiber distance on $\{t\} \times \mathbb{R}$ as the induced path metric. Since any quasi-isometry quasi-preserves the vertex spaces, it quasi-preserves the fiber distance. In terms of the $\mathbb{R}$ coordinate this distance is just $e^{-h(t)}|x_1 - x_2|$.

For any two t and t' in T, let $p : \{t\} \times \mathbb{R} \to \{t'\} \times \mathbb{R}$ be closest point projection. We define the fiber distortion of p as:

$$\frac{d_F(p(x), p(y))}{d_F(x, y)}$$

where x and y are any points on the vertex space over t, and d_F is the fiber distance.

This distortion is $e^{h(t)-h(t')}$. Closest point projection between the vertex spaces is preserved by the quasi-isometry, to within a distance determined by $d(t,t')$. As we let $|x-y|$ go to infinity this additive constant has less and less effect on the distortion. Thus the limit of distortion of points farther and farther apart is bounded above and below by multiples, depending only on the quasi-isometry constants of F, of $e^{h(t)-h(t')}$. This shows that the height change $h(t)-h(t')$ differs from $h(f(t))-h(f(t'))$ by at most some uniform additive error. This is precisely the definition of coarsely orientation preserving.

$\square$

Thus any quasi-isometry F of X covers an almost orientation preserving quasi-isometry f of T. According to the results of §1, $f \times Id$ is a quasi-isometry of X. The quasi-isometry constants of $f \times Id$ may be much larger than those of F. Even if F was an isometry, $f \times Id$ need not be. There is an extension which is better. If f is any coarsely orientation preserving map then define the height change of f, $h(f)$, as the height change between t and $f(t)$ for some t in T. This change is defined up to an error determined by the C in the definition of coarsely orientation preserving. The map $f \times e^{-h(f)}$ is a quasi-isometry with constants that depend only on the quasi-isometry and coarsely orientation preserving constants of f.

The lemma shows that the group of quasi-isometries of X splits as a semi-direct product of the group of coarsely orientation preserving quasi-isometries of T and those quasi-isometries of X which lie over the identity on T. In the case of $BS(1,n)$ we can identify the coarsely orientation preserving quasi-isometries as $Bilip(\mathbb{Q}_n)$ and the quasi-isometries covering the identity as $Bilip(\mathbb{R}$. In this case the full quasi-isometry group is the product of the two. The situation is more complicated in the case of $BS(m,n)$.

A quasi-isometry, F, covering the identity takes each vertex space to itself. For each t in T, let $f_t : \mathbb{R} \to \mathbb{R}$ be the restriction of the quasi-isometry to the vertex space $\{t\} \times \mathbb{R}$. If F is an (A,B) quasi-isometry then there are some (A',B') for which F restricted to each vertex space is an (A',B') quasi-isometry with respect to fiber distance. The fiber distance between (t,x) and (t,y) is $e^{-h(t)}|x-y|$, so f_t is an $(A', B'e^{h(t)}$ quasi-isometry.

Let e be an edge of T with endpoints t and t', where $h(t) \geq h(t')$. The distance between (t,x) and (t',y) is $1+e^{-h}|x-y|$, and the distance between $(t, f_t(x))$ and $(t', f_{t'}(y))$ is $1+e^{-h}|f_t(x)-f_{t'}(y)|$. Given that f_t and $f_{t'}$ are quasi-isometries with constants as above, F will be an (A',B') quasi-isometry on the strip $e \times \mathbb{R}$ if and only if $|f_t(x) - f_{t'}(x)| \leq e^h(A'+B'-1)$.

In summary, F is a quasi-isometry covering the identity if and only if, for some A, B, and C, for each t in T, f_t is an $(A, Be^{h(t)})$ quasi-isometry and, for each edge in e in T, we have $d(f_t, f_{t'}) < Ce^h$.

Consider the metric d^l on T where each edge has length e^h, where h is the height of the higher endpoint. We define the *lower boundary*, $\partial^l T$ as the ideal points of the metric completion of T with respect to this metric. The previous paragraph shows that a quasi-isometry covering the identity is a Lipschitz map from (T, d^l) to $QI(\mathbb{R})$ so that the map f_t is an (A, Be^h) quasi-isometry. This gives a Lipschitz map from ∂^l to $Bilip(\mathbb{R})$ so that the image has uniformly bounded Lipschitz constants.

Lemma 4.2. *Let $Bilip_L(\mathbb{R})$ be the space of bilipschitz maps with bilipschitz constant at most L, equipped with the metric $d(f,g) = |f-g|_\infty + |f^{-1} - g^{-1}|_\infty$. The quasi-isometries of X covering the identity on T are the bilipschitz maps $\partial^l T \to Bilip(\mathbb{R})$ which are contained in $Bilip_L$ for some L for some L.*

Proof. We saw above that any quasi-isometry of X induces such a map from $\partial^l T$ to $Bilip(\mathbb{R})$. So we need to see that any such map extends to a quasi-isometry of X, and that this extension is unique up to bounded distance.

For any t in T, the distance from t to $\partial^l T$ in the metric d^l is bounded above and below by multiples of e^h. Let F and F' induce the same maps on $\partial^l T$. For any t in T, pick a in $\partial^l T$ at minimal distance. We must have a constant K so that $d(F_t, F_a) \leq Ke^{h(t)}$ and the same for F'_t. So $d(F_t, F'_t) \leq 2Ke^{h(t)}$. As the distance along the vertex space over t is scaled by $e^{-h(t)}$, this shows F and F' are at bounded distance.

Essentially the same argument allows us to construct an extension. Given a map on $\partial^l T$, and any t in T, we pick any a in $\partial^l T$ at minimal distance from t and define f_t to be equal to f_a. For t and t' in T, and any a and a' in $\partial^l T$ at minimal distance from them, we know that $d^l(a,a') \leq Ke^{h(t)} + Ke^{h(t')} + d^l(t,t')$. Since the map on the lower boundary is M Lipschitz

$$d(f_a, f_{a'}) \leq Md^l(a,a') \leq MK(e^{h(t)} + e^{h(t')}) + Md^l(t,t')$$

So long as t and t' are not equal, $d(t,t') \geq e^{max(h(t),h(t'))}$ so we have:

$$d(f_a, f_{a'}) \leq M(2K+1)d^l(t,t')$$

Thus the extension is a quasi-isometry T to T. □

We can express this bilipschitz map from $\partial^l T$ to $Bilip(\mathbb{R})$ differently: it can all be assembled into a single bilipschitz map $\partial^l T \times \mathbb{R}$ to itself which covers the identity map of $\partial^l T$.

Any coarsely orientation preserving quasi-isometry of T induces a bilipschitz map of $\partial^l T$. If T has at least two edges decreasing height at each vertex, $\partial^l T$ is dense in the boundary of T, so the map on $\partial^l T$ determines the quasi-isometry up to bounded distance.

We have proven:

Theorem 4.3. *Let T be the Bass-Serre tree of $BS(m,n)$ for $1 < m < n$. The group of coarsely orientation preserving of T is a subgroup, G, of*

$Bilip(\partial^l T)$, and the group of quasi-isometries of $BS(m,n)$ is the group of bilipschitz bundle maps of $\partial^l T \times \mathbb{R}$ covering G.

It would be nice to understand which bilipschitz maps of the lower boundary come from coarsely orientation preserving quasi-isometries. It seems likely that some sort of conformal structure should do the trick.

There is also an upper boundary, defined as the limit points of T with edges scaled by e^{-h}. An coarsely orientation preserving quasi-isometry of T also induces a bilipschitz map of this upper boundary. As with the lower boundary, this boundary is typically dense in the full boundary and so a quasi-isometry is determined by its action on the upper boundary. In the case of $BS(1,n)$ this boundary is $\mathbb{Q}_n$ and the bilipschitz group of the upper boundary is exactly the group of coarsely orientation preserving quasi-isometries.

5. Other Applications

The results of [MSW] show that any group quasi-isometric to a graph of $\mathbb{Z}$s is a graph of virtual $\mathbb{Z}$s. Graphs of virtual $\mathbb{Z}$s have models like those of §1, except that the vertex spaces are only quasi-isometric to $\mathbb{Z}$ rather than isomorphic to $\mathbb{Z}$. This is all that we use about the vertex spaces, so our results apply in this slightly greater generality.

Theorem 5.1. *Let Γ be a finitely generated group. Γ is quasi-isometric to $BS(2,3)$ iff Γ is a graph of virtual $\mathbb{Z}$s which is neither commensurable to $F_n \times \mathbb{Z}$ nor virtually solvable.*

More generally, the techniques of this paper can be used to study more general graphs of groups. One certainly needs to assume that the Bass-Serre tress has bounded valence, which means that all of the edge-to-vertex inclusions have finite index image. In this case, all of the edge and vertex groups are commensurable. We call such a graph of groups *homogeneous.*

Very little can be said in general, as one needs to understand the large scale dynamics of isomorphisms among finite index subgroups of the vertex groups. One case where this is possible is graphs of groups in which every vertex and edge groups is $\mathbb{Z}^n$ for some fixed n. The isomorphisms among the finite index subgroups can be represented as elements of $SL_n(\mathbb{Q})$.

In order for the geometry to reduce to coarsely oriented trees, one needs all of these isomorphisms to lie on a single one parameter subgroup of $GL_n(\mathbb{R})$. The natural examples of this type are HNN extensions of Z^n along finite index subgroups. Let G be $\mathbb{Z}^n$, G' and G'' finite index subgroups, and $T : G' \to G''$ an isomorphism. Abstractly, these HNN extensions are the groups:

$$\Gamma_T = < G, t | t^{-1} g t = Tg, \text{ for } g \in G' >$$

We assume that at least one of the groups G' or G'' is a proper subgroup of G. Groups of this type are studied in [FM3]. Recall that the *Absolute Jordan form* of T is the matrix which is the Jordan form of T except that the values on the diagonal are the norms of the eigenvalues rather than the eigenvalues themselves.

Theorem 5.2. [FM3] *Let Γ_T and $\Gamma_{T'}$ be as above.*

- *If Γ_T and $\Gamma_{T'}$ are quasi-isometric, then for some $\alpha \in \mathbb{R}^+$ the absolute Jordan forms of T^α and T' are equal.*
- *If G_T and $G_{T'}$ are solvable (which is equivalent to one of the subgroups nonproper) then G_T and $G_{T'}$ are quasi-isometric iff there is an $\alpha \in \mathbb{Q}^+$ for which the absolute Jordan forms of T^α and T' are equal.*

The results of this paper allow us to complete the classification.

Theorem 5.3. *Let Γ_T and $\Gamma_{T'}$ be as above. If neither is solvable, and the there is an $\alpha \in \mathbb{R}^+$ so that the absolute Jordan forms of T^α and T' are the same, then Γ_T and $\Gamma_{T'}$ are quasi-isometric.*

It is interesting that for the nonsolvable cases one has a complete invariant of the quasi-isometry type, and a continuous family of quasi-isometry types, while in the solvable cases one has a discrete refinement of the invariant.

We hope to explore more general homogeneous graphs of groups, and the nature of their invariants, in future work.

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