Coarse differentiation of quasi-isometries I: spaces not quasi-isometric to Cayley graphs

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Abstract

In this paper, we prove that certain spaces are not quasi-isometric to Cayley graphs of finitely generated groups. In particular, we answer a question of Woess and prove a conjecture of Diestel and Leader by showing that certain homogeneous graphs are not quasi-isometric to a Cayley graph of a finitely generated group.

This paper is the first in a sequence of papers proving results announced in [EFW1]. In particular, this paper contains many steps in the proofs of quasi-isometric rigidity of lattices in Sol and of the quasi-isometry classification of lamplighter groups. The proofs of those results are completed in [EFW2].

The method used here is based on the idea of *coarse differentiation* introduced in [EFW1].

1 Introduction and statements of rigidity results

For any group Γ generated by a subset S one has the associated Cayley graph, $C_\Gamma(S)$. This is the graph with vertex set Γ and edges connecting any pair of elements which differ by right multiplication by a generator. There is a natural Γ action on $C_\Gamma(S)$ by left translation. By giving every edge length one, the Cayley graph can be made into a (geodesic) metric space. The distance on Γ viewed as the vertices of the Cayley graph is the *word metric*, defined via the norm:

$$\|\gamma\| = \inf\{\text{length of a word in the generators } S \text{ representing } \gamma \text{ in } \Gamma.\}$$

Different sets of generators give rise to different metrics and Cayley graphs for a group but one wants these to be equivalent. The natural notion of equivalence in this category is *quasi-isometry*:

Definition 1.1. *Let (X, d_X) and (Y, d_Y) be metric spaces. Given real numbers $\kappa \geq 1$ and $C \geq 0$, a map $f : X \to Y$ is called a (κ, C)-*quasi-isometry *if*

1. $\frac{1}{\kappa} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \kappa d_X(x_1, x_2) + C$ *for all x_1 and x_2 in X, and,*
2. *the C neighborhood of $f(X)$ is all of Y.*

This paper begins the proofs of results announced in [EFW1] by developing the technique of *coarse differentiation* first described there. Proofs of some of the results in [EFW1] are continued in [EFW2]. Even though quasi-isometries have no local structure and conventional derivatives do not make sense, we essentially construct a "coarse derivative" that models the large scale behavior of the quasi-isometry.

A natural question which has arisen in several contexts is whether there exist spaces not quasi-isometric to Cayley graphs. This is uninteresting without some assumption on homogeneity on the space, since Cayley graphs clearly have transitive isometry group. In this paper we prove that two types of spaces are not quasi-isometric to Cayley graphs. The first are non-unimodular three dimensional solvable groups which do not admit left invariant metrics of nonpositive curvature. The second are the Diestel-Leader graphs, homogeneous graphs first constructed in [DL] where it was conjectured that they were not quasi-isometric to any Cayley graph. We prove this conjecture, thereby answering a question raised by Woess in [SW, W].

Our work is also motivated by the program initiated by Gromov to study finitely generated groups up to quasi-isometry [Gr1, Gr2, Gr3]. Much interesting work has been done in this direction, see e.g. [E, EF, FM1, FM2, FM3, FS, KL, MSW, P1, S1, S2, Sh, W]. For a more detailed discussion of history and motivation, see [EFW1].

We state our results for solvable Lie groups first as it requires less discussion:

Theorem 1.2. *Let $\mathrm{Sol}(m, n) = \mathbb{R} \ltimes \mathbb{R}^2$ be a solvable Lie group where the $\mathbb{R}$ action on $\mathbb{R}^2$ is defined by $z \cdot (x, y) = (e^{mz}x, e^{-nz}y)$ for for $m, n \in \mathbb{R}^+$ with $m > n$. Then there is no finitely generated group Γ quasi-isometric to $\mathrm{Sol}(m, n)$.*

If $m > 0$ and $n < 0$, then $\mathrm{Sol}(m, n)$ admits a left invariant metric of negative curvature. The fact that there is no finitely generated group quasi-isometric to G in this case, provided $m \neq n$, is a result of Kleiner [K], see also [P2]. When $m = n$, the group $\mathrm{Sol}(n, n)$ contains cocompact lattices which are (obviously) quasi-isometric to $\mathrm{Sol}(n, n)$. In the sequel to this paper we prove that any group quasi-isometric to

Sol(n, n) is virtually a lattice in Sol(n, n) [EFW2]. Many of the partial results in this paper hold for $m \geq n$ and are used in that paper as well. Note that the assumption $m \geq n$ is only to fix orientation and that the case $m < n$ can be reduced to this one by changing coordinates.

We also obtain the following, which is an immediate corollary of [FM3, Theorem 5.1] and Theorem 2.1 below:

Theorem 1.3. Sol(m, n) *is quasi-isometric to* Sol(m', n') *if and only if* $m'/m = n'/n$.

Before stating the next results, we recall a definition of the Diestel-Leader graphs, DL(m, n). In this setting, $m, n \in \mathbb{Z}^+$ and we assume $m \geq n$. Let T_1 and T_2 be regular trees of valence $m + 1$ and $n + 1$ respectively. Choose orientations on the edges of T_1 and T_2 so each vertex has n (resp. m) edges pointing away from it. This is equivalent to choosing ends on these trees. We can view these orientations as defining height functions f_1 and f_2 on the trees (the Busemann functions for the chosen ends). If one places the point at infinity determining f_1 at the bottom of the page and the point at infinity determining f_2 at the top of the page, then the trees can be drawn as:

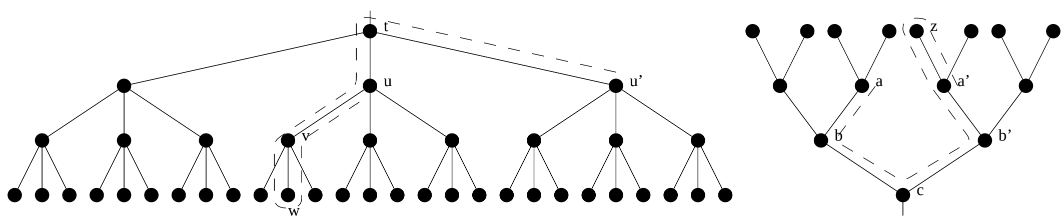

Figure 1: The trees for DL$(3, 2)$. Figure borrowed from [PPS].

The graph DL(m, n) is the subset of the product $T_1 \times T_2$ defined by $f_1 + f_2 = 0$. There is strong analogy with the geometry of solvable groups which is made clear in section 3.

Theorem 1.4. *There is no finitely generated group quasi-isometric to the graph* DL(m, n) *for* $m \neq n$.

For $n = m$ the Diestel-Leader graphs arise as Cayley graphs of lamplighter groups $\mathbb{Z} \wr F$ for $|F| = n$. This observation was apparently first made by R.Moeller and P.Neumann [MN] and is described explicitly, from two slightly different points of view, in [Wo2] and [W]. In [EFW2] we classify lamplighter groups up to quasi-isometry and prove that any group quasi-isometric to a lamplighter group is a lattice in $\mathrm{Isom}(DL(n,n))$ for some n. As discussed above, many of the technical results in this paper are used in those proofs.

We also obtain the following analogue of Theorem 1.3:

Theorem 1.5. *$\mathrm{DL}(m,n)$ is quasi-isometric to $\mathrm{DL}(m',n')$ if and only if m and m' are powers of a common integer, n and n' are powers of a common integer, and $\log m'/\log m = \log n'/\log n$.*

Unlike Theorem 1.3, the proof of this theorem is not completed in this paper. The case when $m = n$ here requires additional arguments. For solvable groups, $\mathrm{Sol}(n,n)$ is always quasi-isometric to $\mathrm{Sol}(n',n')$ for all n and n'. As indicated by the statement of the theorem, this is not true for $DL(n,n)$ and $DL(n',n')$ which are only quasi-isometric when n and n' are powers of a common integer. This last statement is only proven in [EFW2].

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2 Quasi-isometries are height respecting

A typical step in the study of quasi-isometric rigidity of groups is the identification of all quasi-isometries of some space X quasi-isometric to the group, see §7 for more details. For us, the space X is either a solvable Lie group $\mathrm{Sol}(m,n)$ or $\mathrm{DL}(m,n)$. In all of these examples there is a special function $h : X \to \mathbb{R}$ which we call the height function and a foliation of X by level sets of the height function. We will call a quasi-isometry of any of these spaces *height respecting* if it permutes the height level sets to within bounded distance (In [FM4], the term used is horizontal respecting). For technical reasons, it is convenient to consider the more general question of quasi-isometries $\mathrm{Sol}(m,n) \to \mathrm{Sol}(m',n')$.

For $\mathrm{Sol}(m,n)$, the height function is $h(x,y,z) = z$.

Theorem 2.1. *For any $m \geq n > 0$, any (κ,C)-quasi-isometry $\phi : \mathrm{Sol}(m,n) \to \mathrm{Sol}(m',n')$ is within bounded distance of a height respecting quasi-isometry $\hat{\phi}$. Fur-*

thermore, this distance can be taken uniform in (κ, C) and therefore, in particular, $\hat{\phi}$ is a (κ', C')-quasi-isometry where κ', C' depend only on κ and C and on m, n, m', n'.

The proof of Theorem 2.1 is completed in this paper in the case $m > n$. The remaining case, $m = n$, is more difficult and is treated in [EFW2]. Most of the argument here applies in both cases and the only difference occurs at what is labelled "Step II" below.

In fact, Theorem 2.1 can be used to identify the self quasi-isometries of $\mathrm{Sol}(m, n)$ completely. We will need the following definition:

Definition 2.2 (Product Map, Standard Map)**.** A map $\hat{\phi} : \mathrm{Sol}(m, n) \to \mathrm{Sol}(m', n')$ is called a *product map* if it is of the form $(x, y, z) \to (f(x), g(y), q(z))$ or $(x, y, z) \to (g(y), f(x), q(z))$, where f, g and q are functions from $\mathbb{R} \to \mathbb{R}$. A product map $\hat{\phi}$ is called *b-standard* if it is the compostion of an isometry with a map of the form $(x, y, z) \to (f(x), g(y), z)$, where f and g are Bilipshitz with the Bilipshitz constant bounded by b.

It is easy to see that any height-respecting quasi-isometry is at a bounded distance from a standard map, and the standard maps from $\mathrm{Sol}(m, n)$ to $\mathrm{Sol}(m, n)$ form a group which is isomorphic to $(\mathrm{Bilip}(\mathbb{R}) \times \mathrm{Bilip}(\mathbb{R})) \ltimes \mathbb{Z}/2\mathbb{Z}$ when $m = n$ and $(\mathrm{Bilip}(\mathbb{R}) \times \mathrm{Bilip}(\mathbb{R}))$ otherwise. Given a metric space X, one defines $\mathrm{QI}(X)$ to be the group of quasi-isometries of X modulo the subgroup of those at finite distance from the identity. Theorem 2.1 then implies that $\mathrm{QI}(\mathrm{Sol}(m, n)) = (\mathrm{Bilip}(\mathbb{R}) \times \mathrm{Bilip}(\mathbb{R})) \ltimes \mathbb{Z}/2\mathbb{Z}$ when $m = n$ and $(\mathrm{Bilip}(\mathbb{R}) \times \mathrm{Bilip}(\mathbb{R}))$ otherwise. This explicit description was conjectured by Farb and Mosher in the case $m = n$.

Recall that $\mathrm{DL}(m, n)$ is defined as the subset of $T_{m+1} \times T_{n+1}$ where $f_1(x) + f_2(y) = 0$ where f_1 and f_2 are Busemann functions on T_{m+1} and T_{n+1} respectively. We fix the convention that Busemann functions decrease as one moves toward the end from which they are defined. We set $h((x, y)) = f_m(x) = -f_n(y)$ which makes sense exactly on $\mathrm{DL}(m, n) \subset T_{m+1} \times T_{n+1}$. Note that in this choice T_{m+1} branches downwards and T_{n+1} branches upwards. The reader can verify that the level sets of the height function are orbits for a subgroup of $\mathrm{Isom}(DL(m, n))$.

Theorem 2.3. *For any $m \geq n$, any (κ, C)-quasi-isometry φ from $\mathrm{DL}(m, n)$ to $\mathrm{DL}(m', n')$ is within bounded distance of a height respecting quasi-isometry $\hat{\varphi}$. Furthermore, the bound is uniform in κ and C.*

Remark: As above, this is proven here when $m > n$ and in [EFW2] when $m = n$.

The discussion of standard and product maps in the setting of $\mathrm{DL}(m, n)$ is slightly more complicated. We let $\mathbb{Q}_l$ be the l-adic rationals. The complement of a point in the boundary at infinity of T_{l+1} is easily seen to be $\mathbb{Q}_l$. Let x be a point in $\mathbb{Q}_m$ and y a point in $\mathbb{Q}_n$. There is a unique vertical geodesic in $\mathrm{DL}(m,n)$ connecting x to y. To specify a point in $\mathrm{DL}(m,n)$ it suffices to specify x, y and a height z. We will frequently abuse notation by referring to the (x,y,z) coordinate of a point in $\mathrm{DL}(m,n)$ even though this representation is highly non-unique.

Theorem 2.3 can be used to identify the quasi-isometries of $\mathrm{DL}(m,n)$ completely. We need to define product and standard maps as in the case of solvable groups, but there is an additional difficulty introduced by the non-uniqueness of our coordinates. This is that maps of the form $(x,y,z) \to (f(x), g(y), q(z))$, even when one assumes they are quasi-isometries, are not well-defined. Different coordinates for the same points will give rise to different images. We will say a quasi-isometry ψ is *at bounded distance* from a map of the form $(x,y,z) \to (f(x), g(y), q(z))$ if $d(\psi(p), (f(x), g(y), q(z)))$ is uniformly bounded for all points and all choices $p = (x,y,z)$ of coordinates representing each point. It is easy to check that $(x,y,z) \to (f(x), g(y), q(z))$ is defined up to bounded distance if we assume that the resulting map of $\mathrm{DL}(m,n)$ is a quasi-isometry. The bound depends on κ, C, m, n, m' and n'.

Definition 2.4 (Product Map, Standard Map)**.** A map $\hat{\phi} : \mathrm{DL}(m,n) \to \mathrm{DL}(m',n')$ is called a *product map* if it is within bounded distance of the form $(x,y,z) \to (f(x), g(y), q(z))$ or or $(x,y,z) \to (g(y), f(x), q(z))$, where $f : \mathbb{Q}_m \to \mathbb{Q}_{m'}$ (or $\mathbb{Q}_{n'}$), $g : \mathbb{Q}_n \to \mathbb{Q}_{n'}$ (or $\mathbb{Q}_{m'}$) and $q : \mathbb{R} \to \mathbb{R}$. A product map $\hat{\phi}$ is called *b-standard* if it is the compostion of an isometry with a map within bounded distance of one of the form $(x,y,z) \to (f(x), g(y), z)$, where f and g are Bilipshitz with the Bilipshitz constant bounded by b.

Any height-respecting quasi-isometry is at a bounded distance from a standard map, and the standard self maps of $\mathrm{DL}(m,n)$ form a group which is isomorphic to $(\mathrm{Bilip}(\mathbb{Q}_m) \times \mathrm{Bilip}(\mathbb{Q}_n)) \ltimes \mathbb{Z}/2\mathbb{Z}$ when $m = n$ and $(\mathrm{Bilip}(\mathbb{Q}_m) \times \mathrm{Bilip}(\mathbb{Q}_n))$ otherwise. Theorem 2.1 implies that $\mathrm{QI}(\mathrm{DL}(m,n)) = (\mathrm{Bilip}(\mathbb{Q}_m) \times \mathrm{Bilip}(\mathbb{Q}_n))$ unless $m = n$ when $\mathrm{QI}(\mathrm{DL}(m,n)) = (\mathrm{Bilip}(\mathbb{Q}_m) \times \mathrm{Bilip}(\mathbb{Q}_n)) \ltimes \mathbb{Z}/2\mathbb{Z}$.

3 Geometry of $\mathrm{Sol}(m,n)$ and $\mathrm{DL}(m,n)$

In this section we describe the geometry of $\mathrm{Sol}(m,n)$ and $\mathrm{DL}(m,n)$, with emphasis on the geometric facts used in our proofs. In this section we allow the possibility that $m = n$.

3.1 Geodesics, quasi-geodesics and quadrilaterals

The upper half plane model of the hyperbolic plane $\mathbb{H}^2$ is the set $\{(x, \xi) \mid \xi > 0\}$ with the length element $ds^2 = \frac{1}{\xi^2}(dx^2 + d\xi^2)$. If we make the change of variable $z = \log \xi$, we get $\mathbb{R}^2$ with the length element $ds^2 = dz^2 + e^{-z}dx^2$. This is the *log model* of the hyperbolic plane $\mathbb{H}^2$. Note that changing ds^2 to $dz^2 + e^{-mz}dx^2$ we are choosing another metric of constant negative curvature, but changing the value of the curvature. This can be seen by checking that the substitution $z \to \frac{z}{m}, x \to \frac{x}{m}$ is a homothety.

The length element of $\mathrm{Sol}(m, n)$ is:

$$ds^2 = dz^2 + e^{-2mz}dx^2 + e^{2nz}dy^2.$$

Thus planes parallel to the xz plane are hyperbolic planes in the log model. Planes parallel to the yz plane are *upside-down* hyperbolic planes in the log model. When $m \neq n$, these two families of hyperbolic planes have different normalization on the curvature. All of these copies of $\mathbb{H}^2$ are isometrically embedded and totally geodesic.

- We use x, y, z coordinates on $\mathrm{Sol}(m, n)$, with z called the *height*, and x called the *depth*. The planes parallel to the xz plane are right-side up hyperbolic planes (in the log model), and the planes parallel to the yz plane are upside-down hyperbolic planes (also in the log model).

- By "distance", "area" and "volume" we mean these quantities in the $\mathrm{Sol}(m, n)$ metric.

We will refer to lines parallel to the x-axis as x-horocycles, and to lines parallel to the y-axis as y-horocycles. This terminology is justified by the fact that each (x or y)-horocycle is indeed a horocycle in the hyperbolic plane which contains it.

We now turn to a discussion of geodesics and quasi-geodesics in $\mathrm{Sol}(m, n)$. Any geodesic in an $\mathbb{H}^2$ leaf in $\mathrm{Sol}(m, n)$ is a geodesic. There is a special class of geodesics, which we call *vertical geodesics*. These are the geodesics which are of the form $\gamma(t) = (x_0, y_0, t)$ or $\gamma(t) = (x_0, y_0, -t)$. We call the vertical geodesic *upward oriented* in the first case, and *downward oriented* in the second case. In both cases, this is a unit speed parametrization. Each vertical geodesic is a geodesic in two hyperbolic planes, the plane $y = y_0$ and the plane $x = x_0$.

Certain quasi-geodesics in $\mathrm{Sol}(m, n)$ are easy to describe. Given two points (x_0, y_0, t_0) and (x_1, y_1, t_1), there is a geodesic γ_1 in the hyperbolic plane $y = y_0$ that joins (x_0, y_0, t_0) to (x_1, y_0, t_1) and a geodesic γ_2 in the plane $x = x_1$ that joins (x_1, y_0, t_1) to a (x_1, y_1, t_1). It is easy to check that the concatenation of γ_1 and γ_2 is a quasi-geodesic.

In first matching the x coordinates and then matching the y coordinates, we made a choice. It is possible to construct a quasi-geodesic by first matching the y coordinates and then the x coordinates. This immediately shows that any pair of points not contained in a hyperbolic plane in $\mathrm{Sol}(m,n)$ can be joined by two distinct quasi-geodesics which are not close together. This is an aspect of positive curvature. One way to prove that the objects just constructed are quasi-geodesics is to note the following: The pair of projections $\pi_1, \pi_2 : \mathrm{Sol}(m,n) \rightarrow \mathbb{H}^2$ onto the xz and yz coordinate planes can be combined into a quasi-isometric embedding $\pi_1 \times \pi_2 : \mathrm{Sol}(m,n) \rightarrow \mathbb{H}^2 \times \mathbb{H}^2$.

This entire discussion is easily mimicked in $\mathrm{DL}(m,n)$ by replacing geodesics and horocycles in hyperbolic planes with geodesics and horocycles in the corresponding trees. When we want to state a fact that holds both for $\mathrm{Sol}(m,n)$ and $\mathrm{DL}(m,n)$, we refer to the *model space* which we denote by $\mathrm{X}(m,n)$.

We define the upper boundary $\partial^+ X$ as the set of equivalence classes of vertical geodesic rays going up (where two rays are considered equivalent if they are bounded distance apart). The lower boundary $\partial_- X$ is defined similarly. It is easy to see that if $X = \mathrm{Sol}(m,n)$ case, $\partial^+ X \cong \mathbb{R}$ and $\partial_- X \cong \mathbb{R}$. If $X = \mathrm{DL}(m,n)$, then $\partial_- X \cong \mathbb{Q}_m$ and $\partial^+ X \cong \mathbb{Q}_n$. As discussed in Section 2, if $x \in \partial_- X$, $y \in \partial^+ X$ and $z \in \mathbb{R}$, we can define $(x,y,z) \in X$ as the point at height z on the unique vertical geodesic connecting x and y.

Landau asymptotic notation. In the following lemma and throughout the paper, we use the notation $a = O(b)$ to mean that $a < c_1 b$ where c_1 is a constant depending only on the quasi-isometry constants (κ, C) of ϕ and on the model space or spaces (i.e. on m,n, m', n'). We use the notation $a = \Omega(b)$ to mean that $a > c_2 b$, where c_2 depends on the same quantities as c_1. We also use the notation $a \gg b$ and $a \ll b$ with the same meaning of the implied constants.

We state here a key geometric fact used at various steps in the proof.

Lemma 3.1 (Quadrilaterals)**.** *Let $\epsilon > 0$ depending only on m', n'. Suppose $p_1, p_2, q_1, q_2 \in \mathrm{X}(m',n')$ and $\gamma_{ij} : [0, \ell_{ij}] \rightarrow \mathrm{X}(m',n')$ are vertical geodesic segments parametrized by arclength. Suppose $C > 0$ and $0 < D < \epsilon \ell_{ij}$.*

Assume that for $i = 1,2$, $j = 1,2$,

$$d(p_i, \gamma_{ij}(0)) \leq C \qquad \textit{and} \qquad d(q_i, \gamma_{ij}(\ell_{ij})) \leq D,$$

so that γ_{ij} connects the C-neighborhood of p_i to the D-neighborhood of q_j. Further assume that for $i = 1,2$ and all t, $d(\gamma_{i1}(t), \gamma_{i2}) \geq (1/10)t - C$ (so that for each i, the two segments leaving the neighborhood of p_i diverge right away). Then there exists $C_1 = O(C)$ such that exactly one of the following holds:

(a) *All four γ_{ij} are upward oriented, and p_2 is within C_1 of the x-horocycle passing through p_1.*

(b) *All four γ_{ij} are downward oriented, and p_2 is within C_1 of the y-horocycle passing through p_1.*

We think of p_1, p_2, q_1 and q_2 as defining a quadrilateral. The content of the lemma is that any quadrilateral has its four "corners" in pairs that lie essentially along horocycles.

In particular, if we take a quadrilateral with geodesic segments γ_{ij} and with $h(p_1) = h(p_2)$ and $h(q_1) = h(q_2)$ and map it forward under a (κ, C)-quasi-isometry $\phi : \mathrm{X}(m,n) \to \mathrm{X}(m',n')$, and if we would somehow know that ϕ sends each of the four γ_{ij} close to a vertical geodesic, then Lemma 3.1 would imply that ϕ sends the p_i to a pair of points at roughly the same height.

To prove Lemma 3.1, we require a combinatorial lemma.

Lemma 3.2 (Complete Bipartite Graphs)**.** *Let Γ be an oriented graph with four vertices p_1, p_2, q_1, q_2 and four edges, such that there is exactly one edge connecting each p_i to each q_j. Then exactly one of the following is true:*

(i) *All the edges of Γ are from some p_i to some q_j.*

(ii) *All the edges of Γ are from some q_j to some p_i.*

(iii) *There exist two vertices v_1 and v_2 which are connected by two distinct directed paths, and at least of the the v_i is p_1 or p_2.*

Proof. Since there are only 16 possibilities for Γ, one can check directly. One way to organize the check is to let k denote the sum of number of edges outgoing from p_1 and the number of edges outgoing from p_2. If $k = 0$, (ii) holds, and if $k = 4$, then (i) holds. It is easy to check that for $1 \leq k \leq 3$, (iii) holds. $\square$

Proof of Lemma 3.1. Let us assume for the moment that all the geodesics are downward oriented. Let x_{ij}, y_{ij} denote the x and y coordinates of the vertical geodesics γ_{ij}. By the assumptions near p_i we have for $i = 1, 2$, $C_2^{-1} \leq |x_{i1} - x_{i2}| e^{-m'h(p_i)} \leq C_2$, where $C_2 = O(C)$. By the assumptions near q_j we have for $j = 1, 2$, $|x_{1j} - x_{2j}| e^{-m'h(q_j)} \leq D_1$ where $D_1 = O(D)$. Note that since for all i, j, $D < \epsilon \ell_{ij}$, we have $D_1 e^{m'h(q_j)} \ll C_2 e^{m'(h(p_i)}$. These inequalities imply that $e^{-m'(h(p_1) - h(p_2))} = O(C_2)$, and also that $|x_{1j} - x_{2j}| e^{-m'h(p_1)} = O(C_2)$. This proves the lemma under the assumption of downward orientation.

The case where all the vertical geodesics are upward oriented is identical (except that one considers differences in y-coordinates instead).

To reduce to the cases already considered, we apply Lemma 3.2 to the graph Γ consisting of the vertices p_1, p_2, q_1, q_2 with edges the vertical geodesics "almost" connecting them. Suppose that possibility (iii) of Lemma 3.2 holds. Then, we would then have two distinct oriented paths η_1 and η_2 connecting v_1 and v_2. Each η_i is either a vertical geodesics, a concatenation two vertical geodesics, one of which ends near the beginning of the other, or a similar concatenation of three vertical geodesics. In each case it is easy to check that each η_i is close to a vertical geodesic λ_i'. (See Lemma 4.4 for a more general variant of this fact.) But this is a contradiction in view of the divergence assumptions, since any pair of vertical geodesics beginning and ending near the same point are close for their entire length. Thus either (i) or (ii) of Lemma 3.2 holds.

$\square$

3.2 Volume and measure

There is a large difference between the unimodular and nonunimodular examples we consider that has to do with the measures of sets, unimodularity and amenability. In the cases where $m = n$ the spaces we consider are metrically amenable and have unimodular isometry group. When $m \neq n$ the spaces are not metrically amenable and the isometry groups are not unimodular, though the isometry group remains amenable as a group. In particular it is immediately clear that $\mathrm{DL}(n,n)$ cannot be quasi-isometric to $\mathrm{DL}(m,n)$ with $m \neq n$ (since one has metric Fölner sets and the other does not). For the same reason, $\mathrm{Sol}(n,n)$ is not quasi-isometric to $\mathrm{Sol}(m',n')$ with $m' \neq n'$.

The natural volume vol on $\mathrm{DL}(m,n)$ is the counting measure, the natural volume on $\mathrm{Sol}(m,n)$ is $\mathrm{vol} = e^{(n-m)z}dxdydz$. Note that for the unimodular case where $m = n$, the volume on $\mathrm{Sol}(m,n)$ is just the standard volume on $\mathbb{R}^3$. In the case when $m \neq n$ we introduce a new measure. In the case of $\mathrm{Sol}(m,n)$ this is just $\mu = dxdydz$. Note that on z level sets this is a rescaling of vol by a factor of $e^{(n-m)z}$. Analogously on $\mathrm{DL}(m,n)$, we choose a height function $h : \mathrm{DL}(m,n) \to \mathbb{Z}$ and let μ be counting measure times $n^{h(x)}m^{-h(x)}$. Recall that we are assuming that $m \geq n$. The measure μ is also introduced in [BLPS] and is natural for many problems.

We now define certain useful subsets of $\mathrm{Sol}(m,n)$. Let $B(L,\vec{0}) = [-\frac{e^{2mL}}{2}, \frac{e^{2mL}}{2}] \times [-\frac{e^{2nL}}{2}, \frac{e^{2nL}}{2}] \times [-\frac{L}{2}, \frac{L}{2}]$. When $m = n$, then $|B(L,\vec{0})| \approx Le^{2mL}$ and $Area(\partial B(L,\vec{0})) \approx e^{2mL}$, so $B(L)$ is a Fölner set.

To define the analogous object in $\mathrm{DL}(m,n)$, we look at the set of points in $\mathrm{DL}(m,n)$ we fix a basepoint $(\vec{0})$ and a height function h with $h(\vec{0}) = 0$. Let L be an even integer and let $\mathrm{DL}(m,n)_L$ be the $h^{-1}([-\frac{L+1}{2}, \frac{L+1}{2}])$. Then $B(L,\vec{0}))$ is the connected component of $\vec{0}$ in $\mathrm{DL}(m,n)_L$. We are assuming that the top and bottom of the box are midpoints of edges, to guarantee that they have zero measure.

We call $B(L,\vec{0})$ a box of size L centered at the identity. In $\mathrm{Sol}(m,n)$, we define the box of size L centered at a point p by $B(L,p) = T_p B(L,\vec{0})$ where T_p is left translation by p. We frequently omit the center of a box in our notation and write $B(L)$. For the case of $\mathrm{DL}(m,n)$ it is easiest to define the box $B(L,p)$ directly. That is let $\mathrm{DL}(m,n)_{[h(p)-\frac{L+1}{2},h(p)+\frac{L+1}{2}]} = h^{^{-1}}([h(p) - \frac{L+1}{2}, h(p) + \frac{L+1}{2}])$ and let $B(L,p)$ be the connected component of p in $\mathrm{DL}(m,n)_{[h(p)-\frac{L+1}{2},h(p)+\frac{L+1}{2}]}$. It is easy to see that isometries of $\mathrm{DL}(m,n)$ carry boxes to boxes.

We record the following lemma which holds for any model space $\mathrm{X}(m,n)$.

Lemma 3.3. *When $m = n$, the fraction of the volume of $B(L)$ which is within ϵL of the boundary of $B(L)$ is $O(\epsilon)$. In all other cases, this is true for the μ-measure but not the volume.*

We first describe $B(L)$ in the case of $\mathrm{Sol}(m,n)$. In this case, the top of $B(L)$, meaning the set $[-\frac{e^{2mL}}{2}, \frac{e^{2mL}}{2}] \times [-\frac{e^{2nL}}{2}, \frac{e^{2nL}}{2}] \times \{\frac{L}{2}\}$, is not at all square - the sides of this rectangle are horocyclic segments of lengths e^{2mL} and 1 - in other words it is just a small metric neighborhood of a horocycle. Similarly, the bottom is also essentially a horocycle but in the transverse direction. Further, we can connect the 1-neighborhood of any point of the top horocycle to the 1-neighborhood of any point of the bottom horocycle by a vertical geodesic segment, and these segments essentially sweep out the box $B(L)$. This picture is even easier to understand in the Diestel-Leader graphs $\mathrm{DL}(n,n)$, where the boundary of the box is simply the union of the top and bottom "horocycles", and the vertical geodesics in the box form a complete bipartite graph between the two. Thus a box $B(L)$ contains a very large number of quadrilaterals.

3.3 Tiling

The purpose of this subsection is to prove the following lemma.

Lemma 3.4. *Choose $L/R \in \mathbb{Z}$. We can write*

$$B(L) = \bigsqcup_{i \in I} B_i(R) \sqcup \Upsilon \tag{1}$$

where $\mu(\Upsilon) = O(R/L)\mu(B(L))$ and the implied constant depends only on the model space.

In the case when $\mathrm{X}(m,n) = \mathrm{DL}(m,n)$*, then Υ can be chosen to be empty. (This is also possible for* $\mathrm{Sol}(m,n)$ *if e^m and e^n are integers.)*

Remark: We will always refer to a decomposition as in equation (1) as a *tiling* of $B(L)$. We often omit specific reference to the set Υ when discussing tilings.

Proof. For simplicity of notation, we assume $B(L)$ is centered at the origin.

We give the proof first in the case of $\mathrm{DL}(m,n)$ where it is almost trivial. Since $\frac{L}{R} \in \mathbb{Z}$, we can partition $[-\frac{L+1}{2}, \frac{L+1}{2}]$ into subsegments of length R which we label $S_1, \cdots, S_J$ where $J = \frac{L}{R}$. We can then look at the sets $\mathrm{DL}(m,n)_j = h^{-1}(S_j)$. Each connected component of $\mathrm{DL}(m,n)_j$ is clearly a box $B_{j,k}(R)$ of size R. Each $B_{j,k}(R)$ is either entirely inside or entirely outside of $B(L)$. We choose only those k for which $B_{j,k}(R) \subset B(L)$. It is also clear that, after reindexing, we have chosen boxes such that $B(L) = \bigsqcup B_i(R)$.

In $\mathrm{Sol}(m,n)$ the proof is similar, though does not in general give an exact tiling. We simply take the box $B(L)$ and cover it as best possible with boxes of size R. Since $\frac{L}{R} \in \mathbb{Z}$, if we take $B(R, \vec{0})$ and look at translates by $(0,0,Rc)$ for c an integer between $-\frac{L}{R}$ and $\frac{L}{R}$, the resulting boxes are all in $B(L)$. We then take the resulting box $B(R,(0,0,Rc))$ at height k and translate it by vectors of the form $(ae^{mcR}, \vec{b}e^{ncR}, 0)$ where $|a| \leq e^{m(L-(c+1)R)}$ and $|b| \leq e^{n(L-(1-c)R)}$ are integers. This results in boxes $B(R,(a,b,c))$ which we re-index as $B_i(R)$. It is clear that every point not in $\bigsqcup_i B_i(R)$ is within R of the boundary of $B(L)$. Letting $\Upsilon = B(L) - \bigsqcup_i B_i(R)$ we have that $\mu(\Upsilon) < O(R/L)\mu(B(L))$ by Lemma 3.3.

□

4 Step 1

All the results of this section hold for $\mathrm{X}(m,n)$ with $m \geq n$, so in particular for the case $m = n$. The case $m = n$ will be used in the sequel [EFW2]. Also, all results in this section hold for quasi-isometric embeddings, i.e. maps satisfying (1) but not (2) of Definition 1.1. In the this part of the paper, our aim is to prove the following:

Theorem 4.1. *Suppose $\theta > 0$, $\epsilon > 0$. Then there exist constants $0 < \alpha < \beta < \Delta$ (depending on θ, ϵ, κ, C and the model spaces) such that the following holds: Let $\phi : \mathrm{X}(m,n) \to \mathrm{X}(m',n')$ be a (κ, C) quasi-isometry and suppose r_0 is sufficiently* *large (depending on κ, C, θ, ϵ). Then for any $L > \Delta r_0$ and any $B(L)$, there exists R with $\alpha r_0 < R < \beta r_0$ such that if one tiles*

$$B(L) = \bigsqcup_{i\in I} B_i(R)$$

then there exists a subset I_g of I with $\mu(\bigcup_{i\in I_g} B_i(R)) \geq (1-\theta)\mu(B(L))$ so that for any $i \in I_g$ there exists $U_i \subset B_i(R)$ with $\mu(U_i) \geq (1-\theta)\mu(B_i(R))$ and a product map $\hat{\phi}_i : B_i(R) \to \mathrm{X}(m', n')$ such that

$$d(\phi|_{U_i}, \hat{\phi}_i) = O(\epsilon R). \tag{2}$$

Remarks. This theorem says that every sufficiently large box $B(L)$ can be tiled by much smaller boxes $B_i(R)$, and for most (i.e. $1-\theta$ fraction) of the smaller boxes $B_i(R)$ there exist a subset U_i containing $1-\theta$ fraction of the μ-measure of $B_i(R)$ on which the map is a product map, up to error $O(\epsilon R) \ll R$. In the case where $n = m$, the measure of $B_i(R)$ is independent of i, and we have exactly $|I_g| \geq (1-\theta)|I|$. When $m \neq n$, both the number of boxes of size R in a height level set tiling and $\mu(B_i(R))$ are functions of height. We note here that it is possible to apply the proof of Theorem 4.1 simultaneously to a finite collection J of boxes $B_j(L)$ all of the same size and obtain the same conclusions (with the same constants) on most of the boxes in J. As long as $m = n$, by most boxes in J we mean most boxes with the counting measure on J. This observation will be used in [EFW2].

One should note that the number R, and the subset where we control the map, depends on ϕ. Also in Theorem 4.1 there is no assertion that the product maps $\hat{\phi}_i$ on the different boxes $B_i(R)$ match up.

This theorem is in a sense an analogue of Rademacher's theorem that a lipschitz function (or map) is differentiable almost everywhere. The boxes $B_i(R)$ with $i \in I_g$ should be thought of as coarse analogues of points of differentiability.

4.1 Behavior of quasi-geodesics

We begin by discussing some quantative estimates on the behavior of quasi-geodesic segments in $\mathrm{X}(m', n')$ (or equivalently in $\mathrm{X}(m, n)$). Throughout the discussion we assume $\alpha : [0, r] \to \mathrm{X}(m', n')$ is a (κ, C)-quasi-geodesic segment for a fixed choice of (κ, C), i.e. α is a quasi-isometric embedding of $[0, r]$ into $\mathrm{X}(m', n')$. A quasi-isometric embedding is a map that satisfies point (1) in Definition 1.1 but not point (2). All of our quasi-isometric embeddings are assumed to be continuous.

Definition 4.2 (**ϵ-monotone).** *A quasigeodesic segment $\alpha : [0, r] \to \mathrm{X}(m', n')$ is ϵ-monotone if for all $t_1, t_2 \in [0, r]$ with $h(\alpha(t_1)) = h(\alpha(t_2))$ we have $|t_1 - t_2| < \epsilon r$.*

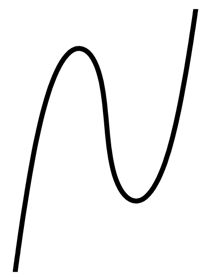

Figure 2: A quasigeodesic segment which is not ϵ-monotone.

In §5 and §6 we will also need a variant. The reader may safely ignore this variant on first reading this section.

Definition 4.3 (Weakly (η, C_1)-monotone)**.** A quasigeodesic segment $\alpha : [0, r] \to \mathrm{X}(m', n')$ is *weakly (η, C_1)-monotone* if for any two points $0 < t_1 < t_2 < r$ with $h(\alpha(t_1)) = h(\alpha(t_2))$, we have $t_2 - t_1 < \eta t_2 + C_1$.

Remark: An ϵ-monotone quasi-geodesic $\alpha : [0, r] \to \mathrm{X}(m', n')$ is a weakly $(\epsilon, \epsilon r)$-monotone quasi-geodesic.

The following fact about ε-monotone geodesics is an easy exercise in hyperbolic geometry:

Lemma 4.4.

(a) *Suppose $\alpha : [0, r] \to \mathrm{X}(m', n')$ is an ϵ-monotone quasi-geodesic segment. Then, there exists a vertical geodesic segment λ in $\mathrm{X}(m', n')$ such that $d(\alpha, \lambda) \leq \omega_1 \epsilon r$, where ω_1 depends only on the model space $\mathrm{X}(m', n')$.*

(b) *Suppose $\alpha : [0, r] \to \mathrm{X}(m', n')$ is a weakly (η, C_1)-monotone quasi-geodesic segment. Then, there exists a vertical geodesic segment λ in $\mathrm{X}(m', n')$ such that $d(\bar{\gamma}(t), \lambda(t)) \leq 2\kappa\eta t + \omega_2 C_1$, where ω_2 depends only on $\mathrm{X}(m', n')$.*

Proof. Both ϵ-monotone and weakly (η, C)-monotone imply that the projections of α onto both xz and yz hyperbolic planes are quasi-geodesics. The result is then a consequence of the Mostow-Morse lemma and the fact that the only geodesics shared by both families of hyperbolic planes are vertical geodesics. One can also prove the lemma by direct computation. ☐

Remark: The distance $d(\alpha, \lambda)$ in (a) is the Hausdorff distance between the sets and does not depend on parametrizations. However, the parametrization on λ implied in (b) is not neccessarily by arc length.

Lemma 4.5 (Subdivision)**.** *Suppose* $\alpha : [0, r] \to \mathrm{X}(m', n')$ *is a quasi-geodesic segment which is not* ϵ*-monotone and* $r \gg C$*. Suppose* $N \gg 1$ *(depending on* ϵ*,* κ*,* C*). Then*

$$\sum_{j=0}^{N-1} \left| h(\alpha(\tfrac{(j+1)r}{N})) - h(\alpha(\tfrac{jr}{N})) \right| \geq |h(\alpha(0)) - h(\alpha(r))| + \frac{\epsilon r}{8\kappa^2}.$$

Informally, the proof amounts to the assertion that if N is sufficiently large, the total variation of the height increases after the subdivision by a term proportional to ϵ. See Figure 3.

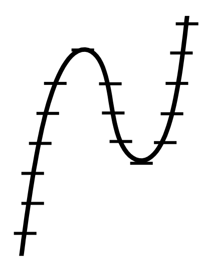

Figure 3: Proof of Lemma 4.5

Proof. Without loss of generality, we may assume that $h(\alpha(0)) \geq h(\alpha(t_1)) = h(\alpha(t_3)) \geq h(\alpha(r))$, where $0 = t_0 < t_1 < t_3 < t_4 = r$ (if not, parametrize in the opposite direction). Since $t_3 - t_1 > \epsilon r$, $\alpha(t_3)$ and $\alpha(t_1)$ are two points in $\mathrm{X}(m', n')$ which are at the same height and are at least $\epsilon r/\kappa$ apart. Then, by $\mathrm{X}(m', n')$ geometry, any quasigeodesic path connecting $\alpha(t_3)$ and $\alpha(t_1)$ must contain a point q such that $|h(q) - h(\alpha(t_1))| \geq (\epsilon r)/(4\kappa)$. Hence, there exists a point t_2 with $t_1 < t_2 < t_3$ such that $|h(\alpha(t_2)) - h(\alpha(t_1))| \geq (\epsilon r)/(4\kappa)$. Hence,

$$\sum_{j=1}^{4} |h(\alpha(t_j)) - h(\alpha(t_{j-1}))| \geq |h(\alpha(0)) - h(\alpha(r))| + \frac{\epsilon r}{4\kappa}$$

If N is large enough then the points t_1, t_2 and t_3 have good approximations of the form jr/N, with $j \in \mathbb{Z}$. This implies the lemma. $\square$

Choosing Scales: Choose $1 \ll r_0 \ll r_1 \ll \cdots \ll r_S$. In particular, $C \ll r_0$ and for $s \in [0, S-1] \cap \mathbb{Z}$, $r_{s+1}/r_s > N$ where N is as in Lemma 4.5.

Lemma 4.6. *Suppose $L \gg r_S$, and suppose $\alpha : [0, L] \to \mathrm{X}(m', n')$ is a quasi-geodesic segment. For each $s \in [1, S]$, subdivide $[0, L]$ into L/r_s segments of length r_s. Let $\delta_s(\alpha)$ denote the fraction of these segments whose images are not ϵ-monotone. Then,*

$$\sum_{s=1}^{S} \delta_s(\alpha) \leq \frac{16\kappa^3}{\epsilon}.$$

Remark: The utility of the lemma is that the right hand side is fixed and does not depend on S. So for S large enough, some (in fact many) $\delta_s(\alpha)$ must be small.

Proof. By applying Lemma 4.5 to each non-ϵ-monotone segment on the scale r_S, we get

$$\sum_{j=1}^{L/r_{S-1}} |h(\alpha(jr_{S-1})) - h(\alpha((j-1)r_{S-1}))| \geq$$
$$\geq \sum_{j=1}^{L/r_S} |h(\alpha(jr_S)) - h(\alpha((j-1)r_S))| + \delta_S(\alpha)\frac{\epsilon L}{8\kappa^2}.$$

Doing this again, we get after S iterations,

$$\sum_{j=1}^{L/r_0} |h(\alpha(jr_0)) - h(\alpha((j-1)r_0))| \geq$$
$$\geq \sum_{j=1}^{L/r_S} |h(\alpha(jr_S)) - h(\alpha((j-1)r_S))| + \frac{\epsilon L}{8\kappa^2}\sum_{s=1}^{S} \delta_s(\alpha).$$

But the left-hand-side is bounded from above by the length and so bounded above by $2\kappa L$. $\square$

4.2 Averaging

In this subsection we apply the estimates from above to images of geodesics under a quasi-isometry from $\mathrm{X}(m, n)$ to $\mathrm{X}(m', n')$. The idea is to average the previous estimates over families of geodesics. In order to unify notation for the two possible model space types, we shift the parametrization of vertical geodesics in $\mathrm{DL}(m,n)$ so that they are parametrized by height minus $\frac{1}{2}$, i.e. by the interval $[-\frac{L}{2}, \frac{L}{2}]$ rather than $[-\frac{L+1}{2}, \frac{L+1}{2}]$.

Setup and Notation.

- Suppose $\phi : \mathrm{X}(m,n) \to \mathrm{X}(m',n')$ is a (κ, C) quasi-isometry. Without loss of generality, we may assume that ϕ is continuous.
- Let $\gamma : [-\frac{L}{2}, \frac{L}{2}] \to \mathrm{X}(m,n)$ be a vertical geodesic segment parametrized by arclength where $L \gg C$.
- Let $\overline{\gamma} = \phi \circ \gamma$. Then $\overline{\gamma} : [-\frac{L}{2}, \frac{L}{2}] \to \mathrm{X}(m',n')$ is a quasi-geodesic segment.

It follows from Lemma 4.6, that for every $\theta > 0$ and every geodesic segment γ, assuming that S is sufficiently large, there exists $s \in [1,S]$ such that $\delta_s(\overline{\gamma}) < \theta$. The difficulty is that s may depend on γ. In our situation, this is overcome as follows:

We will average the result of Lemma 4.6 over Y_L, the set of vertical geodesics in $B(L)$. Let $|Y_L|$ denote the measure/cardinality of Y_L. We will always denote our average by Σ, despite the fact that when $\mathrm{X}(m,n) = \mathrm{Sol}(m,n)$ this is actually an integral over Y_L and not a sum. When $\mathrm{X}(m,n) = \mathrm{DL}(m,n)$ it is actually a sum. Changing order, we get:

$$\sum_{s=1}^{S} \left(\frac{1}{|Y_L|} \sum_{\gamma \in Y_L} \delta_s(\overline{\gamma}) \right) \le \frac{16\kappa^3}{\epsilon}.$$

Let $\delta > 0$ be a small parameter (In fact, we will choose δ so that $\delta^{1/4} = \min(\epsilon, \theta/256)$, where θ is as in Theorem 4.1). Then, if we choose $S > \frac{16\kappa^3}{\epsilon\delta^4}$, then there exists a scale s such that

$$\frac{1}{|Y|} \sum_{\gamma \in Y} \delta_s(\bar{\gamma}) \le \delta^4. \tag{3}$$

Conclusion. On the scale $R \equiv r_s$, at least $1 - \delta^4$ fraction of all vertical geodesic segments of length R in $B(L)$ have nearly vertical images under ϕ.

From now on, we fix this scale, and drop the index s. We will refer to segments of length R arising in our subdivision as *edges* of length R. In the case of $DL(m,n)$ these edges are unions of edges in the graph. In what follows we will use the terms *big edges* for edges of length R if there is any chance of confusion with an actual edge in the graph $DL(m,n)$.

Remark. The difficulty is that, at this point, even though we know that most edges have images under ϕ which are nearly vertical, it is possible that some may have images which are going up, and some may have images which are going down.

4.3 Alignment

We assume that $L/R \in \mathbb{Z}$. As described in §3.3, we tile

$$B(L) = \bigsqcup_{i \in I} B_i(R).$$

Let Y_i denote the set of vertical geodesic segements in $B_i(R)$. We have

$$\frac{1}{|Y_L|} \sum_{\gamma \in Y_L} \delta_s(\bar{\gamma}) = \sum_{i \in I} \frac{\mu(B_i(R))}{\mu(B(L)} \left(\frac{1}{|Y_i|} \sum_{\lambda \in Y_i} \delta_s(\bar{\lambda}) \right) + O(\tfrac{R}{L}), \tag{4}$$

where $\bar{\lambda} = \phi \circ \lambda$, and $\delta(\bar{\lambda}) = \delta_s(\bar{\lambda})$ is equal to 0 if λ is ϵ-monotone, and equal to 1 otherwise. The error term of $O(R/L)$ is due to the fact that the tiling may not be exact, see Lemma 3.4. To justify equation 4, one uses that $\mu(B(L)) = |Y_L|L$ and $\mu(B_i(R)) = |Y_i|R$.

Since the left hand side is bounded by δ^4 and assuming $R/L \ll \delta^2$, we conclude the following:

Lemma 4.7. *Let us tile $B(L)$ by boxes $B_i(R)$ of size R, so that $B(L) = \bigsqcup_{i \in I} B_i(R)$. Then there exists a subset I_g of the indexing set I with $\mu(\bigcup_{i \in I_g} B_i(R)) \geq (1 - \delta^2)\mu(B(L))$ such that if we let Y_i denote the set of vertical geodesics in $B_i(R)$ then*

$$\frac{1}{|Y_i|} \sum_{\gamma \in Y_i} \delta_s(\bar{\gamma}) \leq 2\delta^2. \tag{5}$$

Note that R is the length of one big edge so that the set Y_i of vertical geodesics in $B_i(R)$ consists of big edges connecting the top to the bottom. The equation (5) means that the fraction these edges which are not ϵ-monotone is at most $2\delta^2$.

Notation. In the rest of §4.3 and in §4.4 we fix $i \in I_g$ and drop the index i.

Lemma 4.8 (Alignment)**.** *Let e be an ϵ-monotone big edge of $B(R)$ going from the bottom to the top. We say that e is "upside-down" if $\phi(e)$ is going down, and "right-side-up" if $\phi(e)$ is going up. Then either the fraction of the big edges in $B(R)$ which are upside down or the fraction of the big edges in $B(R)$ which are right-side-up is at least $1 - 4\delta$.*

Proof. We have a natural notion of "top" vertices and "bottom" vertices so that each big edge connects a bottom vertex to a top vertex. Then $B(R)$ is a complete bipartite graph. There must be a subset E of vertices of density $1 - 4\delta$, such that for each vertex in $v \in E$ the fraction of the edges incident to v which are not ϵ-monotone is at most $\delta/2$. Let Γ_1 be the subgraph of $B(R)$ obtained by erasing any edge e such that $\phi(e)$ is not ϵ-monotone. We orient each edge e of Γ_1 by requiring that $\phi(e)$ is going down.

Let $p_1, p_2 \in E$ be any two top vertices. Then we can find two bottom vertices q_1, q_2 such that all four quasigeodesic segments $\phi(\overline{p_1 q_1})$, $\phi(\overline{p_1 q_2})$, $\phi(\overline{p_2 q_1})$ and $\phi(\overline{p_2 q_2})$ are all ϵ-monotone, $\overline{p_1 q_1}$ and $\overline{p_1 q_2}$ diverge quickly at p_1, and $\overline{p_2 q_1}$ and $\overline{p_2 q_2}$ diverge quickly at p_2.

We now apply Lemma 3.1 to conclude that $h(\phi(p_1)) = h(\phi(p_2)) + O(\epsilon R)$ and that all the segments $\phi(\overline{p_i q_j})$ with $i, j = 1, 2$ have the same orientation. Thus any two top vertices in E have images on essentially the same height, say h_1. Similarly, any two bottom vertices in E have images on the same height, say h_2. Since we must have $h_1 > h_2$ or $h_1 < h_2$, the lemma holds. ☐

We define the *dominant orientation* to be right-side-up or upside-down so that the fraction of big edges which have the dominant orientation is at least $1 - 4\delta$.

4.4 Construction of a product map

Let Y' denote the space of pairs (γ, x) where $\gamma \in Y$ is a vertical geodesic in $B(R)$ and $x \in \gamma$ is a point. Let $|\cdot|$ denote uniform measure on Y'. (In the case of $\mathrm{DL}(m, n)$ this is just the counting measure.) The following lemma is a formal statement regarding subsets of Y' of large measure.

Lemma 4.9. *Suppose $R \gg 1/\theta_1$ (where the implied constant depends only on the model space). Suppose $E \subset Y'$, with $|E| \geq (1 - \theta_1)|Y'|$. Then, there exists a subset $U \subset B(R)$ such that:*

(i) $\mu(U) \geq (1 - 2\sqrt{\theta_1})\mu(B(R))$, *where μ is defined in* §3.2.

(ii) *If $x \in U$, then for at least $(1-\sqrt{\theta_1})$ fraction of the vertical geodesics $\gamma \in Y$ passing within distance $1/2$ of x, $(\gamma, x) \in E$.*

Remark. Note that for the case of $\mathrm{DL}(m,n)$, any geodesic passing within distance $(1/2)$ of x passes through x.

Proof. For $x \in B(R)$, let $Y(x) \subset Y$ denote the set of geodesics which pass within $1/2$ of x. For clarity, we first give the proof for the $\mathrm{DL}(m,n)$ case. Note that $|Y(x)| = c\mu(\{x\})$, where c depends on m, n and the location and size of $B(R)$. Note that

$$|Y|R = \sum_{x \in B(R)} \sum_{\gamma \in Y(x)} 1 = \sum_{x \in B(R)} |Y(x)| = \sum_{x \in B(R)} c\mu(\{x\}) = c\mu(B(R)) \tag{6}$$

Suppose $f(\gamma, x)$ is any function of a geodesic γ and a point $x \in \gamma$: then,

$$\begin{aligned}
\frac{1}{|Y|R} \sum_{\gamma \in Y} \sum_{x \in \gamma} f(\gamma, x) &= \frac{1}{|Y|R} \sum_{x \in B(R)} \sum_{\gamma \in Y(x)} f(\gamma, x) \\
&= \frac{1}{|Y|R} \sum_{x \in B(R)} \frac{1}{|Y(x)|} \sum_{\gamma \in Y(x)} |Y(x)| f(\gamma, x) \\
&= \frac{1}{\mu(B(R))} \sum_{x \in B(R)} \frac{1}{|Y(x)|} \sum_{\gamma \in Y(x)} \mu(\{x\}) f(\gamma, x), \qquad (7)
\end{aligned}$$

where in the last line we used (6).

We apply (7) with f the the characteristic function of the complement of E. We get,

$$\frac{1}{\mu(B(R))} \sum_{x \in B(R)} \mu(\{x\}) \left(\frac{1}{|Y(x)|} \sum_{\gamma \in Y(x)} f(\gamma, x) \right) < \theta_1 \tag{8}$$

Let $F(x)$ denote the parenthesized quantity in the above expression. Let $E_2 = \{x \in B(R) \ : \ F(x) > \sqrt{\theta_1}\}$. Recall that Markov's inequality says that for any real-valued function f, and any real number $a > 0$, the measure of the set $\{|f| > a\}$ is at most $\frac{1}{a} \int |f|$. Then, by this inequality, $\mu(E_2)/\mu(B(R)) \leq \sqrt{\theta_1}/\theta_1 = \sqrt{\theta_1}$, and for $x \notin E_2$, for at least $(1-\sqrt{\theta_1})$ fraction of the geodesics γ passing through x, $(\gamma, x) \in E$.

This completes the proof for the $\mathrm{DL}(m,n)$ case. In $\mathrm{Sol}(m,n)$ the computation is essentially the same, except for the fact that $|Y(x)|$ (i.e. the measure of set of geodesics passing within $(1/2)$ of x) can become smaller when x is within $(1/2)$ of the boundary of $B(R)$. However, the relative μ measure of such points is $O(1/R)$ by Lemma 3.3. Therefore (6) and (7) hold up to error $O(1/R) < \theta_1$. $\square$

Corollary 4.10. *There exists a subset $U \subset B(R)$ with $\mu(U) > (1 - 8\sqrt{\delta})\mu(B(R))$ such that for $x \in U$, $(1 - 2\sqrt{\delta})$-fraction of the geodesics passing within $(1/2)$ of x are ϵ-monotone and have the dominant orientation.*

Proof. Let E denote the set of pairs (γ, x) where $\gamma \in Y$ is a dominantly oriented ϵ-monotone geodesic segment, and x is a point of γ. Let $U \subset B(R)$ be the subset constructed by Lemma 4.9. Since $|E| \geq (1 - 4\delta)|Y'|$, $\mu(U) \geq (1 - 8\sqrt{\delta})\mu(B(R))$. $\square$

Lemma 4.11. *Suppose ϕ and $B(R)$ and U are as in Corollary 4.10. Then, there exist functions $\psi : \mathbb{R}^3 \to \mathbb{R}^2$, $q : \mathbb{R} \to \mathbb{R}$, and a subset $U_1 \subset B(R)$ with $\mu(U_1) > (1 - 128\delta^{1/4})\mu(B(R))$ such that for $(x, y, z) \in U_1$,*

$$d(\phi(x, y, z), (\psi(x, y, z), q(z))) = O(\epsilon R) \tag{9}$$

Proof. We assume that the dominant orientation is right-side-up (the other case is identical). Now suppose $p_1, p_2 \in U$ belong to the same x-horocycle. By the construction of U there exist q_1, q_2 in $B(R)$ (above p_1, p_2) such that for each $i = 1, 2$ the two geodesic segments $\overline{p_i q_1}$ and $\overline{p_i q_2}$ leaving p_i diverge quickly, and each of the quasi-geodesic segments $\phi(\overline{p_i q_j})$ is ϵ-monotone. Then by Lemma 4.4, each of the $\phi(\overline{p_i q_j})$ is within $O(\epsilon R)$ of a quasi-geodesic segment λ_{ij}. Now by applying Lemma 3.1 to the λ_{ij} we see that $\phi(p_1)$ and $\phi(p_2)$ are on the same x-horocycle, up to an error of $O(\epsilon R)$. Thus, the restriction of ϕ to U preserves the x-horocycles. A similar argument (but now we will pick q_1, q_2 below p_1, p_2) shows that the restriction of ϕ to U preserves the y-horocycles. We can now conclude that ϕ is height respecting on a slightly smaller set U_1 , i.e. there exist functions $\psi : \mathbb{R}^3 \to \mathbb{R}^2$ and $q : \mathbb{R} \to \mathbb{R}$ such that for $(x, y, z) \in U_1$, (9) holds. $\square$

Proposition 4.12. *Suppose ϕ and $B(R)$ and U are as in Corollary 4.10. Then, there exist functions f, g, q, a corresponding product map $\hat{\phi}$, and a subset $U_2 \subset B(R)$ with $\mu(U_2) > (1 - 256\delta^{1/4})\mu(B(R))$ such that for $(x, y, z) \in U_2$,*

$$d(\phi(x, y, z), \hat{\phi}(x, y, z)) = O(\epsilon R).$$

Proof. To simplify language, we assume that the dominant orientation is right-side-up (the other case is identical). Let z_1 (resp. z_2) denote the height of the bottom (resp. top) of $B(R)$. If $(x, y, z) \in B(R)$, we let $\gamma_{xy} : [z_1, z_2] \to B(R)$ denote the vertical geodesic segement $\gamma_{xy}(t) = (x, y, t)$. Let F_1 (resp. F_2) denote the subset of the bottom (resp. top) face of $B(R)$ which is within $8\delta^{1/4}$ of a point of U. Since

$\mu(U) \geq (1-8\sqrt{\delta})\mu(B(R))$, each F_i has nearly full μ-measure. In fact if we let $U' \subset B(R)$ denote the set of points (x,y,z) such that $(x,y,z_1) \in F_1$, $(x,y,z_2) \in F_2$, and γ_{xy} has ϵ-monotone image under ϕ, then $\mu(U') \geq (1-8\delta^{1/4})\mu(B(R))$.

Note that F_1 is an $O(1)$ neighborhood of a (subset of a) segment of a x-horocycle, say $\{(x,y_1,z_1) \ : \ x \in A\}$. Since the restriction of ϕ to U preserves the x-horocycles, $\delta^{1/4} < \epsilon$, there exist numbers y_1' and z_1' and a function $f: A \to \mathbb{R}$ or $\mathbb{Q}_m$ such that for $x \in A$, $\phi(x,y_1,z_1)$ is at most $O(\epsilon R)$ distance from $(f(x), y_1', z_1')$. Similarly, F_2 is bounded distance from a set of the form $\{(x_2,y,z_2) \ : \ y \in A'\}$, and there exists a function $g: A' \to \mathbb{R}$ or $\mathbb{Q}_n$ such that the restriction of ϕ to F_2 is $O(\epsilon R)$ distance from a map of the form $(x_2,y,z_2) \to (x_2', g(y), z_2')$.

Let U_1 be as in Lemma 4.11. Now suppose $p = (x,y,z) \in U' \cap U_1$. Since $p \in U'$, $\phi(p)$ is $O(\epsilon R)$ from a vertical geodesic connecting a point in the $O(\epsilon R)$ neighborhood of $\phi(x,y,z_1)$ to a point in the $O(\epsilon R)$ neighborhood of $\phi(x,y,z_2)$. Hence, $\phi(p)$ is within $O(\epsilon R)$ distance of the vertical geodesic connecting $(f(x), y_1', z_1')$ to $(x_2', g(y), z_2')$. This, combined with (9) implies the proposition, and hence Theorem 4.1. $\square$

Remark: The product map $\hat{\phi}$ produced in the proof of Proposition 4.12 is not defined on the entire box. Since we are not assuming anything about the regularity of the maps f, g and q which define $\hat{\phi}$, one can choose an arbitrary extension to a product map defined on the box. This is sufficient for our purposes here.

Order in which constants are chosen.

- We may assume that ϵ is sufficiently small so that in Lemma 3.1, the $O(\epsilon r)$ error term is smaller then $(r/100)$.
- We choose $N = N(\epsilon, \kappa, C)$, so that Lemma 4.5 works. We may assume $N \in \mathbb{Z}$.
- As described in §4.2, we choose $\delta = \delta(\epsilon, \theta, \kappa, C)$, so that $\delta^{1/4} < \epsilon$ (see proof of Proposition 4.12), $256\delta^{1/4} < \theta$ (see Proposition 4.12) and also $\delta^2 < \theta$, see Lemma 4.7.
- We choose $S = S(\delta, \kappa, \epsilon)$ so that $S > \frac{32\kappa^3}{\epsilon\delta^4}$, as described in §4.2.
- For $s = 1, \ldots, S$, write $r_s = r_0 N^s$.
- Write $L = \Delta r_0$. Choose $\Delta = N^p$ for some $p \in \mathbb{Z}$, and so that for $R = r_S$, the $O(R/L)$ error term in (4) is at most δ^2. Then the same is true for any $R = r_s$, $1 \leq s \leq S$.

Now assume r_0 is sufficiently large so that Lemma 4.9 holds with $R = r_0$ and $\theta_1 = 2\delta$. Theorem 4.1 holds with $\alpha = 1$ and $\beta = N^S$.

5 Step II

In this section, we assume that $m > n$. We prove the following theorem:

Theorem 5.1. *For every $\delta > 0$, $\kappa > 1$ and $C > 0$ there exists a constant $L_0 > 0$ (depending on δ, κ, C) such that the following holds: Suppose $\phi : \mathrm{X}(m,n) \to \mathrm{X}(m',n')$ is a (κ, C) quasi-isometry. Then for every $L > L_0$ and every box $B(L)$, there exists a subset $U \subset B(L)$ with $|U| \geq (1-\delta)|B(L)|$ and a height-respecting map $\hat{\phi}(x,y,z) = (\psi(x,y,z), q(z))$ such that*

(i)
$$d(\phi|_U, \hat{\phi}) = O(\delta L).$$

(ii) For z_1, z_2 heights of two points in $B(L)$, we have
$$\frac{1}{2\kappa}|z_1 - z_2| - O(\delta L) < |q(z_1) - q(z_2)| \leq 2\kappa |z_1 - z_2| + O(\delta L). \tag{10}$$

(iii) For all $x \in U$, at least $(1-\delta)$ fraction of the vertical geodesics passing within $O(1)$ of x are $(\eta, O(\delta L))$-weakly monotone.

Remark. It is not difficult to conclude from Theorem 5.1 that $\hat{\phi}$ is in fact a product map (not merely height-respecting). However, we will not need this.

Theorem 5.1 is true also for the case $m = n$; its proof for that case is the content of [EFW2]. The proof presented in this section is much simpler, but applies only to the case $m > n$. The main point of the proof is to show that if $m > n$, then in the notation of Theorem 4.1, for each $i \in I_g$, the maps $\hat{\phi}_i$ must preserve the up direction. This is done in §5.3. The deduction of Theorem 5.1 from that fact is in §5.4.

5.1 Volume estimates

The following is a basic property shared for example by all homogeneous spaces and all spaces with a transitive isometry group (such as $\mathrm{X}(m,n)$).

Lemma 5.2. *For $p \in \mathrm{X}(m,n)$, let $D(p,r)$ denote the metric ball of radius r centered at p. Then, for every $b > a > 0$ there exists $\omega = \omega(a,b) > 1$ with $\log \omega = O(b-a)$ such that for all $p, q \in \mathrm{X}(m,n)$,*
$$\omega(a,b)^{-1}|D(p,a)| \leq |D(q,b)| \leq \omega(a,b)|D(p,a)|,$$
where $|\cdot|$ denotes volume (relative to the $\mathrm{X}(m,n)$ metric). Also $\log \omega(a,b) = O(b-a)$, where the implied constant depends on the model space $\mathrm{X}(m,n)$.

Proof. The first statement is immediate since $\mathrm{X}(m,n)$ is a homogeneous space. The second is a simple calculation. $\square$

In this section we prove some fairly elementary facts about quasi-isometries quasi-preserving volume. The main tool is the following basic covering lemma.

Lemma 5.3. *Let X be a metric space and let $\mathcal{F}$ be a collection of points in X. Then for any $a > 0$ there is a subset $\mathcal{G}$ in $\mathcal{F}$ such that:*

(i) *The sets $\{B(x,a) | x \in \mathcal{G}\}$ are pairwise disjoint.*

(ii) $\bigcup_{\mathcal{F}} B(x,a) \subset \bigcup_{\mathcal{G}} B(x,5a)$.

This lemma and it's proof (which consists of picking $\mathcal{G}$ by a greedy algorithm) can be found in [He, Chapter 1]. This argument is implicit in almost any reference which discusses covering lemmas.

Recall that we are assuming that ϕ is a continuous (κ, C) quasi-isometry.

From this we can deduce the following fact about quasi-isometries of $\mathrm{X}(m,n)$. This fact holds much more generally for metric measure spaces which satisfy Lemma 5.2.

Proposition 5.4. *Let $\phi : \mathrm{X}(m,n) \to \mathrm{X}(m',n')$ be a continuous (κ, C) quasi-isometry. Then for any $a \gg C$ there exists $\omega_1 > 1$ with $\log \omega_1 = O(a)$ such that for any $U \subset \mathrm{X}(m,n)$,*

$$\omega_1^{-1} |\phi(N_a(U))| \leq |N_a(U)| \leq \omega_1 |N_a(\phi(U))|$$

where $N_a(U) = \{x \in \mathrm{X}(m,n) \ : \ d(x,U) < a\}$.

Proof. We assume that $a > 4\kappa C$. Note that we are assuming that every point is within distance C of the image of ϕ. Let $\mathcal{F}$ be the covering of $N_a(U)$ consisting of all balls of radius a centered in U. By Lemma 5.3, we can find a (finite) subset $\mathcal{G}$ of U such that $\bigcup_{x \in \mathcal{G}} D(x,5a)$ cover $N_a(U)$ and such that the balls centered at $\mathcal{G}$ are pairwise disjoint. Hence,

$$\sum_{x \in \mathcal{G}} |D(x,a)| \leq |N_a(U)| \leq \sum_{x \in \mathcal{G}} |D(x,5a)|.$$

Now $\phi(N_a(U))$ is covered by $\bigcup_{x \in \mathcal{G}} \phi(D(x,5a)) \subset \bigcup_{x \in \mathcal{G}} D(\phi(x), 5\kappa a + C)$. Hence,

$$|\phi(N_a(U))| \leq \sum_{x \in \mathcal{G}} |D(\phi(x), 5\kappa a + C)| \leq \omega(a, 5\kappa a + C) \sum_{x \in \mathcal{G}} |D(x,a)| \leq \omega(a, 5\kappa a + C) |N_a(U)|.$$

For the other inequality,

$$|N_a(\phi(U))| \geq |N_a(\phi(\mathcal{G}))| = \left|\bigcup_{x\in\mathcal{G}} D(\phi(x),a)\right| \geq \left|\bigcup_{x\in\mathcal{G}} D(\phi(x),a/\kappa - C)\right|$$
$$= \sum_{x\in\mathcal{G}} |D(\phi(x),a/\kappa - C)| \geq \omega(a/\kappa - 2C, 5a)^{-1} \sum_{x\in G} |D(x,5a)| \geq \omega_1^{-1}|N_a(U)|$$

□

Terminology. The "coarse volume" of a set E means the volume of $N_a(E)$ for a suitable a. If the set E is essentially one dimensional (resp. two dimensional) we use the term "coarse length" (resp. "coarse area") instead of coarse volume but the meaning is still the volume of $N_a(E)$. We also use $\ell(\cdot)$ to denote coarse length.

5.2 The trapping lemma

For a path γ, let $\ell(\gamma)$ denote the length of γ (measured in the $\mathrm{X}(m,n)$-metric).

Recall that we are assuming $m \geq n$.

Lemma 5.5. *Suppose L is a constant z plane, and suppose U is a bounded set contained in L. Suppose $k > r > 0$ and γ is a path which stays at least k units below L, i.e. that $\max_{x\in\gamma}(h(x)) < h(L) - k$. Suppose also that any vertical geodesic ray starting at U and going down intersects the r-neighborhood of γ. Then,*

$$\ell(\gamma) \geq e^{c_1 k - c_2 r}\,\mathrm{Area}(U)$$

where $c_1 > 0$ and $c_2 > 0$ depend only on the model space, and both the length and the area are measured using the $\mathrm{X}(m,n)$ metric.

Proof. We give a proof for $DL(m,n)$, the proof for $\mathrm{Sol}(m,n)$ is similar. Let Δ denote the r-neighborhood of γ, then $|\Delta| \leq e^{c_2 r}\ell(\gamma)$. Pick N so that Δ stays above height $h(L) - N$. Let $\mathcal{A}$ denote the set of vertical segments of length N which start at height $h(L)$ and go down. Let $\mathcal{A}_U$ denote the elements of $\mathcal{A}$ which start at points of U. Then $|\mathcal{A}_U| = |U|m^N$. Now for $0 < s < N$, any point at height $h(U) - s$ intersects exactly $n^s m^{N-s}$ elements of $\mathcal{A}$. Thus, by the assumption on the height of Δ, any point of Δ can intersect at most $n^k m^{N-k}$ elements of $\mathcal{A}$. But by assumption, Δ intersects any element of $\mathcal{A}_U$. Thus, $|\Delta| \geq (|U|m^N)/(n^k m^{N-k}) = |U|(m/n)^k$, which implies the lemma. (Recall that in our notation, length = area = cardinality in $DL(m,n)$).

$\square$

Remark: When $m = n$, Lemma 5.5 still holds (but with $c_1 = 0$), and also in addition with the word *below* replaced by the word *above*. When $m \neq n$, volume decreases on upwards projection.

5.3 Vertical Orientation preserved

Given a box $B(R)$, an y-horocycle H in $B(R)$, and a number ρ, we let the *shadow*, $\mathrm{Sh}(H,\rho)$, of H in $B(R)$ be the set of points that can be reached by a vertical geodesics going straight down from the ρ neighborhood of H. Note that if H is the top of the box, $\mathrm{Sh}(H,1)$ is the entire box. Similar definitions hold for x-horocycles, but then the shadow will be above the horocycle.

The goal of this subsection is the following:

Theorem 5.6. *Suppose $m > n$ and that ϵ and θ are sufficiently small (depending only on the model space). Let I, I_g, U_i and $\hat{\phi}_i$ be as in Theorem 4.1. Suppose $i \in I_g$. Then the product map $\hat{\phi}_i : B(R) \rightarrow \mathrm{X}(m', n')$ can be written as $\hat{\phi}_i(x, y, z) = (f_i(x), g_i(y), q_i(z))$, with $q_i : \mathbb{R} \to \mathbb{R}$ coarsely orientation preserving.*

Remark. The result of Theorem 5.6 is *false* in the case $m = n$, since there exist "flips", i.e. isometries which reverse vertical orientation. This is the point where the proof in the case $m = n$ diverges from the case $m > n$.

In the rest of §5.3 we prove Theorem 5.6. We pick $i \in I_g$ and suppress the index i for the rest of this subsection.

Pick $1 \gg \rho_2 \gg \rho_1 \gg \epsilon$ to be determined later (see the end of this subsection).

Lemma 5.7. *All but $O(4\sqrt{\theta})$ proportion of the y-horocycles H that are above the middle of the box $B(R)$ have all but $O(\sqrt{\theta})$ fraction of the μ-measure of both $N_{\rho_1 R}(H)$ and $\mathrm{Sh}(H, \rho_1 R)$ in U.*

Proof. Let P be a constant z plane above the middle of the box (and not too close to the top). We choose horocycles H_i in P such that $P = \coprod_i N_{\rho_1 R}(H_i)$. The subset of $B(R)$ below P is then the disjoint union of the shadows $\coprod_i \mathrm{Sh}(H_i, \rho_1 R)$. Since at least half the measure of $B(R)$ is below P, it follows that there is some i so that all but $O(\sqrt{\theta})$ of the μ-measure of $\mathrm{Sh}(H_i, \rho_1 R)$ is in U. To guarantee the same fact about $N_{\rho_1 R}(H)$, we pick P such that $N_{\rho_1 R}(P)$ has all but $O(\sqrt{\theta})$ fraction of its μ-measure in U. $\square$

Lemma 5.8. *For any H as in Lemma 5.7, there exists a constant z plane P such that $N_{\rho_1 R}(P) \cap \mathrm{Sh}(H, \rho_1 R) \cap U$ contains all but $O(\theta^{1/4})$ fraction of the μ-measure in $N_{\rho_1 R}(P) \cap \mathrm{Sh}(H, \rho_1 R)$. Furthermore, we can choose P and H such that $\rho_2 R < d(P, H) < 2\rho_2 R$.*

Proof. Let

$$E = \mathrm{Sh}(H, \rho_1 R) \cap h^{-1}(h(H) - 2\rho_2 R, h(H) - \rho_2 R).$$

By Lemma 5.7, $\mu(E \cap U) \geq (1 - c\sqrt{\theta}/\rho_2)\mu(E) \geq (1 - \theta^{1/4})\mu(E)$, where c is the implied constant in Lemma 5.7, and we have assumed that $c\theta^{1/2}/\rho_2 \leq \theta^{1/4}$. Now this is another application of Fubini's theorem, where we partition E into its intersections with neighborhood of constant z planes. $\square$

Lemma 5.9. *Let P be as in the conclusion of Lemma 5.8. There are subsets S_1, S_2 of $P \cap B(R)$ such that*

1. *$N_{\rho_1 R}(S_i) \cap U$ contains all but $O(\theta^{1/4})$ fraction of the μ-measure of $N_{\rho_1 R}(S_i)$.*
2. *for $s_i \in S_i$ any path joining s_1 to s_2 of length less than $\kappa^3 \rho_2 R$ passes within $O(\rho_1 R)$ of H.*
3. *For $i = 1, 2$, $\mathrm{Area}(S_i) \gg \frac{1}{6} \mathrm{Area}(\mathrm{Sh}(H, \rho_1 R) \cap P) > e^{c\rho_2 R} \ell(H \cap B(R))$ where c depends on the model spaces.*

Proof. We divide $P \cap \mathrm{Sh}(H, \rho_1 R)$ into equal thirds where each third has the entire y-extent and a third of the x-extent. We let $\tilde{S}_1$ and $\tilde{S}_2$ be the two non-middle thirds. Now let S_i be the portion of $\tilde{S}_i$ which is at least $\kappa^3 \rho_2 R$ away from the edges of $B(R)$. The area of each of these regions is much more than the coarse area $\ell(H \cap B(R))$ since projecting upwards decreases area and each region projects upwards onto H. $\square$

The proof of Theorem 5.6 involves deriving contradiction to the reversal of orientation of vertical geodesics under $\hat{\phi}$ on $B_i(R)$. The goal is to show that if the orientation were to reverse, we could find a path in the target joining $\hat{\phi}(s_1)$ to $\hat{\phi}(s_2)$ that contradicts Lemma 5.9.

Proof of Theorem 5.6. We assume that vertical orientation is not preserved but reversed. This means the z component $q(z)$ of the product map in Theorem 4.1 is orientation reversing. Let H_p denote the x-horocycle through p. For $i = 1, 2$, let

$$S_i' = \{ p \in S_i \cap U \quad : \quad \text{for } j = 1, 2, \ \ell(H_p \cap U \cap S_j) > 0.5 \ell(H_p \cap S_j) \ \}$$

Let $W_i = \hat{\phi}(S_i)$. By Proposition 5.4, and Lemma 5.9 part 3, we have

$$\text{Area}(W_i) \geq e^{-D\rho_1 R}\,\text{Area}(S_i') > e^{(c\rho_2 - D\rho_1)R}\ell(H) \tag{11}$$

where D depends only on the model spaces, κ and C. Assuming ϵ is sufficiently small, $\ell(H) \ll \text{Area}(W_i)$. Then by Lemma 5.5, 99.9% of the geodesics going down from W_i do not enter the $O(\rho_1 R)$ neighborhood of $\phi(H)$. Let W_i' denote the set of $p \in W_i$ so that 99% of the geodesics going down from (the 1/2-neighborhood of) p do not enter the $O(\rho_1 R)$ neighborhood of $\phi(H)$. Then $\text{Area}(W_i') \geq 0.9\,\text{Area}(W_i)$.

Let $\mathcal{H}$ denote the set of y-horocycles H' such that $W_i \cap H'$ is non-empty for some (or equivalently for all) $i \in \{1, 2\}$. Let $\mathcal{H}_i = \{H' \in H \;:\; W_i' \cap H' \neq \emptyset\}$. We claim that $\mathcal{H}_1 \cap \mathcal{H}_2 \neq \emptyset$. Indeed, $W_i = \coprod_{H' \in H} W_i \cap H'$, and $\text{Area}(W_i) = |\mathcal{H}|c_i$, where $c_i = |W_i \cap H'|$ is independent of $H' \in \mathcal{H}$. Now,

$$0.9c_i|\mathcal{H}| = 0.9\,\text{Area}(W_i) \leq \text{Area}(W_i') \leq c_i|\mathcal{H}_i|.$$

Thus, we have $|\mathcal{H}_i| \geq 0.9|\mathcal{H}|$, and hence there exists $H' \in \mathcal{H}_1 \cap \mathcal{H}_2$. By the definition of the $\mathcal{H}_i$, we can find p_1, p_2 such that $p_i \in H' \cap W_i'$. Then for $i = 1, 2$, by the definition of W_i', we can find geodesics γ_i going down from p_i, such that γ_i do not enter the $O(\rho_1 R)$ neighborhood of $\phi(H)$, and such that γ_1 and γ_2 meet at some point p'. By construction $d(p', W_i) \leq 2\kappa^2\rho_2 R$. Concatenating subsegments of these two geodesics yields a path connecting p_1 to p_2 of length $d(p_1, p_2)$ which avoids the $O(\rho_1 R)$ neighborhood of $\phi(H)$. Pulling back, we have a path of length at most $16\kappa^3\rho_2 R$ and avoiding the $O(\rho_1 R)$ neighborhood of H, which connects a point within $O(\epsilon R)$ of S_1 to a point within $O(\epsilon R)$ of S_2. This contradicts Lemma 5.9. ☐

Remark: The only place in this paper where we make essential use of the fact that ϕ satisfies (2) of Definition 1.1 is in pulling back the path connecting p_1 and p_2 at the end of the proof of Theorem 5.6.

Choice of constants. Let A be the largest constant depending only on κ, C and the model spaces, which arises in the course of the argument in §5.3.

We choose ρ_2 so that $A\rho_2 < 1$. Similarly we choose ρ_1 so that $A\rho_1 < \rho_2$ and ϵ $A\epsilon < \rho_1$. We also choose θ so that $A\theta^{1/4} < \rho_1$. We also make sure that ϵ and θ are sufficiently small so that Theorem 5.6 applies. In addition we choose r_0 in the statement of Theorem 4.1 such that the constant $e^{(c\rho_2 - D\rho_1)R}$ that appears in the proof of Theorem 5.6 is at least 1000. Our other choices guarantee that $\rho_2 > \frac{D}{c}\rho_1$, so this is just a lower bound on R and therefore r_0.

5.4 Proof of Theorem 5.1

The uniform set and the exceptional set. Let I_g and U_i, $i \in I_g$ be as in Theorem 4.1. Let $W = \bigcup_{i\in I_g} U_i$.

Recall that Y_L is the set of vertical geodesics in $B(L)$. Here we will work with a fixed geodesic $\gamma \in Y_L$. Let $W^c \subset B(L)$ denote the complement of W in $B(L)$. For a point $x \in \gamma$ and $T > 0$ let

$$P(x, \gamma, T) = |W^c \cap \gamma \cap D(x, T)|,$$

where $D(x, T)$ is the ball of radius T centered at x (so that $\gamma \cap D(x, T)$ is an interval of length $2T$ centered at x).

Lemma 5.10. *For every $\eta_1 > 0$ there exists $\eta > 0$ (with $\eta \to 0$ as $\eta_1 \to 0$) such that the following holds: Suppose γ is a geodesic ray leaving x, and for any $T > 1$, $P(x, \gamma, T) < \eta_1 T$. Then, $\bar{\gamma} = \phi \circ \gamma$ is (η, C_1)-weakly-monotone, where $C_1 = O(\eta_1 R)$.*

Proof. Parametrize γ so that $\gamma(0) = x$. Without loss of generality, we may assume that γ is going up. Let $\bar{\gamma} = \phi \circ \gamma$. Suppose $0 < t_1 < t_2$ are such that $h(\bar{\gamma}(t_1)) = h(\bar{\gamma}(t_2))$. Write $q(t) = h(\bar{\gamma}(t)$. Subdivide $[t_1, t_2]$ into intervals $I_1, \dots, I_N$ of length $\leq \eta_1 R$ and so that the length of all but the first and last is exactly $\eta_1 R$. We may assume $N \geq 3$. Let $J \subset [1, \dots, N]$ be the set of $j \in \mathbb{Z}$ such that $\bar{\gamma}(I_j) \cap W \neq \emptyset$. For $j \in J$, pick s_j such that $\bar{\gamma}(s_j) \in W$, and pick $s \in I_j$ arbitrarily otherwise. Now

$$0 = q(t_2) - q(t_1) = q(t_2) - q(s_{N'}) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{N'} (q(s_j) - q(s_{j-2})) + q(s_1) - q(t_1), \tag{12}$$

where N' is either N or $N - 1$ depending on whether N is odd or even.

Let $Q_0 = \{\text{odd } j \in [3, N'] \;\; : \;\; \gamma(s_j) \in U_i, \; \gamma(s_{j-2}) \in U_i\}$ (same U_i). Let Q_1 denote the set of odd $j \in [3, N']$ such that $\gamma(s_j)$ and $\gamma(s_{j+1})$ are in different boxes $B_i(R)$. Finally, let Q_2 denote the set of odd $j \in [3, N']$ such that $\gamma(I_j) \subset W^c$ or $\gamma(I_{j-2}) \subset W^c$. By assumption, $|Q_2| \leq t_2/R$ and also $|Q_1| \leq t_2/R$. Then, $|Q| \geq (1/3)(t_2 - t_1)/(\eta_1 R)$. Note that if $j \in Q$, then $|q(s_j) - q(s_{j-2})| \geq \eta_1 R/(2\kappa)$, and for any j, $|q(s_{j+2}) - q(s_j)| \leq 4\kappa\eta_1 R$. Hence,

$$\sum_{j\in Q} q(s_j) - q(s_{j-2}) \geq |Q| \frac{\eta_1 R}{\kappa} \geq \frac{t_2 - t_1}{6\kappa}.$$

Also,

$$\left|\sum_{j\in Q_1\cup Q_2} q(s_j)-q(s_{j-2})\right| \le |Q_1\cup Q_2|2\kappa\eta_1 R \le 2\kappa\eta_1 t_2$$

Plugging into (12) we see that

$$0 \ge \frac{t_2-t_1}{6\kappa} - 2\kappa\eta_1 t_2 - O(\eta_1 R),$$

or

$$\frac{t_2-t_1}{6\kappa} \le 2\kappa\eta_1 t_2 + O(\eta_1 R)$$

which implies the lemma. □

Pick $A \gg 1$ (in fact we will eventually choose $A = (4(128/\delta)^4)$. Suppose $\gamma \in Y_L$, We define a point $x \in \gamma$ to be *A-uniform along* γ, if for all $T > 1$,

$$\frac{P(x,\gamma,T)}{T} < A\frac{|\gamma\cap W^c|}{L}$$

Lemma 5.11. *Let $\theta(\gamma)$ denote the proportion of non-A-uniform points along γ. Then, $\theta(\gamma) \le 2/A$.*

Proof. This is a standard application of the Vitali covering lemma. Let $\nu = \frac{|\gamma\cap W^c|}{L}$. Suppose x is non-uniform, then there is an interval I_x centered at x such that

$$|I_x \cap W^c| \ge A\nu|I_x|.$$

The intervals I_x obviously cover the non-uniform set of γ, and, by Vitali, we can choose a disjoint subset I_j which cover at least half the measure of the non-uniform set. Then,

$$|\bigcup I_j| \le \sum |I_j| \le (A\nu)^{-1}|I_j\cap W^c| \le (A\nu)^{-1}|\gamma\cap W^c|$$

Dividing both sides by L (the length γ), and recalling that $|\gamma\cap W^c|/L = \nu$, we obtain the estimate. □

Let $\theta_1 = \frac{\theta}{\eta_1} + \frac{2}{A}$.

Corollary 5.12. *There exists a subset $U \subset B(L)$ with $\mu(U) > (1-2\sqrt{\theta_1})\mu(B(L))$ such that for $x \in U$, $(1-\sqrt{\theta_1})$-fraction of the geodesics passing within $(1/2)$ of x are right-side-up $(\eta,\eta_1 R)$-weakly-monotone.*

Proof. Let Y' denote the space of pairs (γ, x) where $\gamma \in Y_L$ is a vertical geodesic in $B(L)$ and $x \in \gamma$ is a point. Let $E \subset Y'$ denote the set of pairs (γ, x) such that $|\gamma \cap W| \geq (1 - \eta_1/A)L$, and x is A-uniform along γ. Then, by Lemma 5.11 we have $|E| \geq (1 - \theta_1)|Y'|$. Let U be the subset constructed by applying Lemma 4.9. Then $\mu(U) \geq (1 - 2\sqrt{\theta_1})\mu(B(L))$, and for $x \in U$ by Lemma 5.10 at least $(1 - \sqrt{\theta_1})$ fraction of the geodesic rays leaving x are $(\eta, O(\eta_1 R))$-weakly-monotone. $\square$

Lemma 5.13. *Suppose ϕ and $B(L)$ and U are as in Corollary 5.12, and η is sufficiently small (depending only on the model space). Then, there exist functions ψ, q, and a subset $U_1 \subset B(L)$ with $\mu(U_1) > (1 - 128\theta_1^{1/4})\mu(B(L))$ such that for $(x, y, z) \in U_1$,*

$$d(\phi(x, y, z), (\psi(x, y, z), q(z))) = O(\delta L) \tag{13}$$

Proof. This proof is identical to that of Lemma 4.11. $\square$

Proof of Theorem 5.1. Choose η so that Lemma 5.13 holds. Choose η_1 so that Lemma 5.10 holds, and also that the $O(\eta_1 R)$ term in Lemma 5.10 is at most δL. Choose $A^{-1} = (\delta/128)^4/4$ and choose $\theta = (\delta/128)^4/\eta_1$ so that $128\theta_1^4 < \delta$. Now the theorem follows from combining Corollary 5.12 and Lemma 5.13. $\square$

6 Step III

In this section, we complete the proof of Theorem 2.1 and Theorem 2.3. We assume that ϕ is a κ, C quasi-isometry from $\mathrm{X}(m, n)$ to $\mathrm{X}(m', n')$ satisfying the conclusion of Theorem 5.1. All the arguments in this section are valid also in the case $m = n$ (and are used in [EFW2]).

6.1 A weak version of height preservation

In this subsection, our main goal is to prove the following:

Theorem 6.1. *Let $\phi : \mathrm{X}(m, n) \to \mathrm{X}(m', n')$ be a (κ, C) quasi-isometry satisfying the conclusions of Theorem 5.1. Then for any $\theta \ll 1$ there exists $M > 0$ (depending on θ, κ, C) such that for any x and y in $\mathrm{X}(m, n)$ with $h(x) = h(y)$,*

$$|h(\phi(x)) - h(\phi(y))| \leq \theta d(x, y) + M. \tag{14}$$

Note. This is a step forward, since the theorem asserts that (14) holds *for all* pairs x, y of equal height (and not just on a set of large measure).

We would like to restrict Theorem 5.1 to the neighborhood of a constant z plane. Let $\nu = \sqrt{\delta}$. Fix a constant z plane P. For notational convinience, assume that P is at height 0. Let $R(L) \subset P$ denote the intersection of P with a box $B(2L)$ whose top face is at height L and bottom face at $-L$. Then $R(L)$ is a rectangle (in fact, when $m = n$, with this choice of P, $R(L)$ is a square in the euclidean metric). We will call L the *size* of $R(L)$. Let $R^+(L)$ denote the "thickening" of $R(L)$ in the z-direction by the amount νL, i.e. $R^+(L)$ is the intersection of $B(2L)$ with the region $\{p \in \mathrm{X}(m,n) \;:\; -(\nu/2)L \leq h(p) \leq (\nu/2)L\}$, where as above $h(\cdot)$ denotes the height function.

We now have the following corollary of Theorem 5.1:

Corollary 6.2. *Suppose $L > L_0$. Then for every rectangle $R(L) \subset P$ there exists $U \subset R^+(L)$ with $\mu(U) \geq (1-\nu)\mu(R^+(L))$ and a standard map $\hat{\phi} : U \to \mathrm{X}(m', n')$ such that $d(\phi|_U, \hat{\phi}) \leq \nu L$. Furthermore, for any $p \in U$, for 99% of the geodesics γ leaving p, $\phi(\gamma \cap B(2L))$ is within δL of a vertical geodesic segment (in the right direction).*

The tilings. Choose $\beta \ll 1$ depending only on κ, C, m and n. When $m = n$, $\beta \approx \frac{1}{\kappa^4}$. Let $L_j = (1+\beta)^j L_0$. For each $j > 0$ we tile P by rectangles R of size L_j; we denote the rectangles by $R_{j,k}$, $k \in \mathbb{N}$. For $x \in \mathrm{X}(m,n)$, let $R_j[x]$ denote the unique rectangle in the j'th tiling to which the orthogonal projection of x to P belongs.

Warning. Despite the fact that $L_{j+1} = (1+\beta)L_j$, the number of rectangles of the form $R_j[x]$ needed to cover a rectangle of the form $R_{j+1}[y]$ is very large (on the order of $e^{\beta L_j}$). This is because the Euclidean size of $R(L_j)$ is approximately e^{L_j}.

The sets U_j. For each rectangle $R_{j,k}$, Corollary 6.2 gives us a subset of $R^+_{j,k}$ which we will denote by $U_{j,k}$. Let

$$U_j = \bigcup_{k=1}^{\infty} U_{j,k}.$$

In view of Corollary 6.2, for any $x \in U_j$,

$$\sup_{y \in R_j^+[x] \cap U_j} |h(\phi(y)) - h(\phi(x))| \leq 2\nu L_j \tag{15}$$

We also have the following generalization:

Lemma 6.3. *For any $x \in U_j$ and any $y \in R^+_{j+1}[x] \cap U_j$,*

$$|h(\phi(y)) - h(\phi(x))| \leq 12\nu L_j.$$

Proof. Let $R_j[p]$ be a rectangle on the same "row" as $R_j[x]$ and the same "column" as $R_j[y]$. Then, since $\nu \ll 1$, there exists an x-horocycle H which intersects both $R^+_j[x] \cap U_j$ and $R^+_j[p] \cap U_j$; let us denote the points of intersection by x_1 and p_1 respectively.

Now for $i = 1, 2$ choose (sufficiently different) vertical geodesics γ_i coming down from (near) x_1 and γ'_i coming down from (near) p_1 such that for $i = 1, 2$, $\gamma_i(L_{j+1})$ and $\gamma'_i(L_{j+1})$ are close. (here all the geodesics are parametrized by arclength). In view of Corollary 6.2, since x_1 and p_1 are in U_j, we may assume that there exist vertical geodesics λ_i and λ'_i such that for $0 \leq t \leq L_j$, $d(\gamma_i(t), \lambda_i) \leq \nu + \eta t$ where $\eta \ll 1$. Similarly, $d(\gamma'_i(t), \lambda'_i) \leq \nu + \eta t$.

Thus, in particular, $h(\phi(\gamma_i(L_j))) \leq h(\phi(x_1)) - L_j/\kappa + \eta \leq h(H) - L_j/(2\kappa)$, and similarly $h(\phi(\gamma'_i(L_j))) \leq h(p_1) - L_j/(2\kappa)$. Now note that $d(\gamma_i(L_j), \gamma'_i(L_j) = \beta L_j + O(1)$. Hence $d(\phi(\gamma_i(L_j)), \phi(\gamma'_i(L_j))) \leq 2\kappa\beta L_j + O(1)$, and by assumption $\kappa^2\beta \ll 1$. Then by Lemma 3.1, $\phi(x_1)$ and $\phi(p_1)$ are near the same horocycle, and thus, in particular,

$$|h(\phi(x_1)) - h(\phi(p_1))| \leq 4\nu L_j \tag{16}$$

Similarly, we can find $p_2 \in R^+_j[p] \cap U_j$ and $y_2 \in R^+_j[y] \cap U_j$ such that p_2 and y_2 are on the same y-horocycle. Then, by the same argument,

$$|h(\phi(p_2)) - h(\phi(y_2))| \leq 4\nu L_j \tag{17}$$

Hence, in view of (16), (17), and (15),

$$|h(\phi(x)) - h(\phi(y))| \leq 12\nu L_j,$$

as required. $\square$

Lemma 6.4. *Suppose $p \in R^+_j[x] \cap U_j$, $q \in R^+_{j+1}[x] \cap U_{j+1}$. Then,*

$$|h(\phi(p)) - h(\phi(q))| \leq 16\nu L_{j+1} \tag{18}$$

Proof. Note that the orthogonal projection of $U_j \cap R^+_{j+1}[x]$ to $R_{j+1}[x]$ has full μ-measure (up to order ν). The same is true of $U_{j+1} \cap R^+_{j+1}[x]$. Thus, the projections intersect, and thus we can find $p' \in U_j \cap R^+_{j+1}[x]$ and $q' \in R^+_{j+1}[x] \cap U_{j+1}$ such that $d(p', q') \leq 2\nu L_{j+1}$. Now, in view of Lemma 6.3,

$$|h(\phi(p)) - h(\phi(p'))| \leq 12\nu L_j$$

and in view of (15),

$$|h(\phi(q')) - h(\phi(q))| \leq 2\nu L_{j+1}$$

This implies (18). $\square$

Proof of Theorem 6.1. We have

$$R_0[x] \subset R_1[x] \subset R_2[x] \subset \dots$$

and

$$R_0[y] \subset R_1[y] \subset R_2[y] \subset \dots$$

There exists N with L_N comparable to $d(x, y)$ such that (after possibly shifting the N'th grid by a bit) $R_N[x] = R_N[y]$. Now for $0 \leq j \leq N$, pick $x_j \in R^+_j[x] \cap U_j$, $y_j \in R^+_j[y] \cap U_j$. We may assume that $x_N = y_N$. Now, using Lemma 6.4,

$$\begin{aligned}
|h(\phi(x_0)) - h(\phi(y_0))| &\leq \sum_{j=0}^{N-1} |h(\phi(x_{j+1}) - h(\phi(x_j))| + \sum_{j=0}^{N-1} |h(\phi(y_{j+1}) - h(\phi(y_j))| \\
&\leq 2 \sum_{j=0}^{N-1} 16\nu L_{j+1} \\
&\leq \frac{32\nu}{\beta} L_N,
\end{aligned}$$

where in the last line we used that $L_j = (1+\beta)^j L_0$. Now since $x_0 \in R_0[x]$, $d(x, x_0) \leq L_0$, so $|h(\phi(x) - \phi(x_0)| = O(L_0)$. Similarly, $|h(\phi(y) - h(\phi(y_0))| = O(L_0)$. Also note that L_{N+1} is within a factor of 2 of $d(x, y)$. Thus the theorem follows. $\square$

6.2 Completion of the proof of height preservation

Lemma 6.5. *Let $\phi : \mathrm{X}(m, n) \to \mathrm{X}(m', n')$ be a (κ, C) quasi-isometry. Then for any $\eta \ll 1$ there exists $C_1 > 0$ (depending on η, κ, C) such that for any vertical geodesic ray γ, $\phi \circ \gamma$ is (η, C_1)-weakly monotone.*

Proof. This is a corollary of Theorem 6.1. Suppose γ is a vertical geodesic ray parametrized by arclength, and $\bar{\gamma} = \phi \circ \gamma$. Suppose $0 < t_1 < t_2$ are such that $h(\bar{\gamma}(t_1)) = h(\bar{\gamma}(t_2))$. We now apply Theorem 6.1 to ϕ^{-1} instead of ϕ (with $x = \bar{\gamma}(t_1)$) and $y = \bar{\gamma}(t_2)$). We get $|h(\gamma(t_1)) - h(\gamma(t_2))| \leq \theta d(\bar{\gamma}(t_1)), \bar{\gamma}(t_2)) + O(M)$, i.e.

$$|t_2 - t_1| \leq \theta\kappa^2|t_2 - t_1| + O(M)$$

I.e. $\bar{\gamma}$ is $(\theta\kappa^2, O(M))$-weakly monotone. □

Proof of Theorem 2.1 and Theorem 2.3. Suppose p_1 and p_2 are two points of $\mathrm{X}(m,n)$, with $h(p_1) = h(p_2)$. We can find q_1, q_2 in $\mathrm{X}(m,n)$ such that p_1, p_2, q_1, q_2 form a quadrilateral. By Lemma 6.5, each of the segments γ_{ij} connecting a point in the $O(1)$ neighborhood of p_i to a point in the $O(1)$ neighborhood of q_j maps under ϕ to an $O(\eta, C_1)$-weakly monotone quasi-geodesic segment. Then by Lemma 4.4, and Lemma 3.1, we see that $h(\phi(p_1)) = h(\phi(p_2)) + O(C_1)$. □

7 Deduction of rigidity results

The purpose of this section is to apply the previous results on self quasi-isometries of $\mathrm{Sol}(m,n)$ and the DL-graphs to understand all finitely generated groups quasi-isometric to either one. This follows a standard outline: if Γ is quasi-isometric to X then Γ quasi-acts on X (in this case that just means there is a homomorphism $\Gamma \to \mathrm{QI}(X)$ with uniformly bounded constants). We then need to show that such a quasi-action can be conjugated to an isometric action. The basic ingredients we need to do this are the following:

Theorem 7.1. *[FM2] Every uniform quasi-similarity action on $\mathbb{R}$ is bilipschitz conjugate to a similarity action.*

The proof of this theorem makes substantial use of work of Hinkannen [H] who had shown that a uniform quasi-symmetric action was quasi-symmetrically conjugate to a symmetric action.

Theorem 7.2. *[MSW] Let Γ have a uniform quasi-similarity action on $\mathbb{Q}_m$. If the Γ action is cocompact on the space of pairs of distinct points in $\mathbb{Q}_m$ then there is some n and a similarity action of Γ on $\mathbb{Q}_n$ which is bilipschitz conjugate to the given quasi-similarity action.*

It is useful to think about these results in a quasi-action interpretation. One can view $\mathbb{R}$ as $S^1 - \{pt\}$, and interpret a uniform quasi-similarity action on $\mathbb{R}$ as the boundary of a quasi-action on $\mathbb{H}^2$ fixing a point at infinity. The result of Farb and Mosher then says that this quasi-action is quasi-conjugate to an isometric action on $\mathbb{H}^2$. The interpretation of the second result is similar, with a tree of valence $m+1$ replacing $\mathbb{H}^2$. The hypothesis of cocompactness on pairs in that theorem then translates to cocompactness of the quasi-action on the tree.

We now state and prove a result that immediately implies Theorem 1.2. This result is also used in [EFW2].

Theorem 7.3. *Assume every (κ, C) self quasi-isometry of* $\mathrm{Sol}(m,n)$ *is at bounded distance from a b-standard map where $b = b(\kappa, C)$. Then any uniform group of quasi-isometries of* $\mathrm{Sol}(m,n)$ *is virtually a lattice in* $\mathrm{Sol}(m,n)$.

Proof. Let $f : \Gamma \to \mathrm{Sol}(m,n)$ be a quasi-isometry. For each γ in Γ we have the self-quasi-isometry T_γ of $\mathrm{Sol}(m,n)$ given by

$$x \mapsto f(\gamma f^{-1}(x))$$

By Theorem 2.1, T_γ is bounded distance from a standard map. On a subgroup Γ' of Γ of index at most two, this gives a homomorphism $\Phi : \Gamma' \to \mathrm{Qsim}(\mathbb{R}) \times \mathrm{Qsim}(\mathbb{R})$. By Theorem 7.1, each of these quasi-similarity actions on $\mathbb{R}$ can be bilipschitz conjugated to a similarity action. This gives $\Psi : \Gamma' \to \mathrm{Sim}(\mathbb{R}) \times \mathrm{Sim}(\mathbb{R})$.

Since the quasi-isometries T_γ have uniformly bounded constants, we know that the stretch factors of the two quasi-similarity actions Φ are approximately on the curve (e^{mt}, e^{-nt}) - meaning that the products weighted by these factors are uniformly close to 1. This therefore is true for Ψ as well. So, in the sequence:

$$\Gamma' \to Sim(\mathbb{R}) \times Sim(\mathbb{R}) \to \mathbb{R} \times \mathbb{R}$$

where the final map is the log of the stretch factor, we know that the image lies within a bounded neighborhood of the line $ny = -mx$. Since the image is a subgroup, this implies it must lie on this line. Since the subgroup of $\mathrm{Sim}(\mathbb{R}) \times \mathrm{Sim}(\mathbb{R})$ above this line is $\mathrm{Sol}(m,n)$, we have produced a homorphism

$$\Psi : \Gamma' \to \mathrm{Sol}(m,n)$$

We now show that the kernel is finite and the image discrete and cocompact. This follows essentially from the fact that the map f is a quasi-isometry.

Consider a compact subset $K \subset \mathrm{Sol}(m,n)$. The set $F = \Psi^{-1}(K)$ consists of maps with uniformly bounded stretch factors, and which move the origin at most a bounded amount. Transporting this information back to the standard maps of $\mathrm{Sol}(m,n)$, we see that for $\gamma \in F$ the maps T_γ move the identity a uniformly bounded amount. However, the quasi-action T of Γ on $\mathrm{Sol}(m,n)$ is the f-conjugate of the left action of Γ on Γ. This action is proper, so we conclude that F is finite. This implies that Ψ has finite kernel and discrete image. In the same way, the fact that the Γ action on Γ is transitive implies that the image of Ψ is cocompact.

Thus the image of Γ' is a lattice in $\mathrm{Sol}(m,n)$.

□

This proves Theorem 1.2, since if $m \neq n$, the group $\mathrm{Sol}(m,n)$ is not unimodular and therefore does not contain lattices.

We next prove Theorem 1.4. In fact, we show

Theorem 7.4. *Assume every (κ, C) self quasi-isometry of $\mathrm{DL}(m,n)$ is at bounded distance from a b-standard map where $b = b(\kappa, C)$. Then any uniform group of quasi-isometries of $\mathrm{DL}(m,n)$ is virtually a lattice in $\mathrm{Isom}(\mathrm{DL}(n',n'))$ where n', m, n are all powers of a common integer.*

Some complications arise from the differences between Theorem 7.2 and Theorem 7.1. We need the following theorem of Cooper:

Theorem 7.5. *[Co] The metric spaces $\mathbb{Q}_p$ and $\mathbb{Q}_q$ are bilipschitz equivalent if and only if there are integers d, s, t so that $p = d^s$ and $q = d^t$.*

This immediately implies a weaker version Theorem 1.5. We now turn to theorem 7.4.

Proof. We proceed as in the previous proof for $\mathrm{Sol}(m,n)$. The first difference is that to apply theorem 7.2 we need to know that the quasi-similarity actions of Γ' on $\mathbb{Q}_n$ and $\mathbb{Q}_m$ are cocompact on pairs of points. As discussed above, this is equivalent to asking the corresponding quasi-action on the trees of valence $n+1$ and $m+1$ to be cocompact. This then follows immediately from the fact that Γ' is cocompact on $\mathrm{DL}(m,n)$.

Thus we have $\Psi : \Gamma' \to \mathrm{Sim}(\mathbb{Q}_a) \times \mathrm{Sim}(\mathbb{Q}_b)$ for some a and b. Thus we know that we have d_i, s_i, t_i for $i = 1, 2$ with $n = {d_1}^{s_1}$, $m = {d_2}^{s_2}$ and:

$$\Psi : \Gamma' \to \mathrm{Sim}(\mathbb{Q}_{{d_1}^{t_1}}) \times \mathrm{Sim}(\mathbb{Q}_{{d_2}^{t_2}})$$

We know, as before, that the weighted stretch factors are approximate inverses. In this case the stretch factors are in $\mathbb{Z}$ - in $Sim(\mathbb{Q}_m)$ one can stretch only by powers of m.

Thus the image is a subgroup lying on the line $\{(a,b); a*\log d_1*\frac{t_1}{s_1}+b*\log d_2*\frac{t_2}{s_2}=0\}$. For this to be a non-empty subgroup of $\mathbb{Z}^2$ we must have $\frac{\log d_1}{\log d_2}$ rational, which implies that there is a d with $d_1 = d^u$, $d_2 = d^v$ for some u and v.

There is still some ambiguity in the choices, since many groups occur as subgroups of $\mathrm{Sim}(\mathbb{Q}_{p^k})$ for many different k. As in the construction of [MSW] we can make the choices unique by choosing the t_i the maximum possible, so that all powers of ${d_i}^{t_i}$ occur as stretch factors. With these choices we are forced to have the line $\{(a,b) : a+b=0\}$ as this is the only line of negative slope in $\mathbb{Z}^2$ surjecting to both factors. Thus we have $\Psi : \Gamma' \to \mathrm{Sim}(\mathbb{Q}_{d^{t_1}}) \times \mathrm{Sim}(\mathbb{Q}_{d^{t_2}})$, with the image contained in the subgroup having inverse stretch factors. This group is, up to finite index, $\mathrm{Isom}(\mathrm{DL}(d^{t_1}, d^{t_2})$. So we have:

$$\Psi : \Gamma' \to \mathrm{Isom}(\mathrm{DL}(d^{t_1}, d^{t_2})$$

Exactly as before, one can see that the kernel is finite and the image is a lattice, which implies that $t_1 = t_2$. This implies that Γ is amenable, and hence it and $\mathrm{DL}(m,n)$ have metric Fölner sets. This is true only for $m = n$, which completes the proof. $\square$

This immediately implies Theorem 1.4, since $\mathrm{DL}(m,n)$ is only amenable as a metric space when $m = n$.

Proof of Theorem 1.3. Since all $\mathrm{Sol}(n,n)$ are obviously quasi-isometric to one another, it suffices to consider the case $m \neq n$. This then follows immediately from Theorem 2.1 and [FM3, Theorem 5.1]. $\square$

Proposition 7.6. *Theorem 2.3 implies Theorem 1.5.*

Proof. In view of Theorem 2.3, the proof of this result is similar to the last one. The point is that (up to permuting m and n) the quasi-isometry $\mathrm{DL}(m,n) \to \mathrm{DL}(m',n')$ induces quasi-similarities $\mathbb{Q}_n \to \mathbb{Q}_{n'}$ and $\mathbb{Q}_m \to \mathbb{Q}_{m'}$. Theorem 7.5 then implies that m and m' are both powers of some number d and that n and n' are both powers of some number s. However, since the quasi-similarities both come from the same map on vertical geodesics, the scale factors must agree. This immediately implies $\log m'/\log m = \log n'/\log n$. $\square$

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