MATH 215, FALL 2018 (WHYTE) MIDTERM SOLUTIONS

(1) $(A \cap B) \cup C = A \cap (B \cup C)$

False. One quick way to see this is the the left hand side contains all of C (since it is something union with C) and the right hand side is a subset of A (since it is something intersect A). Thus is there are elements in C that are not in A the two sides will not be equal. For example, if $C = \{x\}$ and A is the empty set then then the left hand side is $\{x\}$ and the right hand side is empty, so they are not equal.

(2) $(A^c \cup B^c) = (A \cup B)^c$

False. If there is an element in A that is not in B then it will be in $A \cup B$ and so not in $(A \cup B)^c$ but is in B^c and so also in $A^c \cup B^c$. So, for example, if $A = \{x, y\}$ and $B = \{y, z\}$ then x is in $A^c \cup B^c$ but not in $(A \cup B)^c$.

(3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

True. I'll prove this by showing each is a subset of the other:

To see $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$, start with $x \in A \cup (B \cap C)$. By the definition of union, this means either $x \in A$ or $x \in B \cap C$. In the first case, $x \in A$, then x is in both $A \cup B$ and $A \cup C$ since these sets are both A union other things - so x is in the intersection of $A \cup B$ and $A \cup C$ as needed. Otherwise, if $x \in B \cap C$ then, by the definition of intersection, x is in both B and C. But $x \in B$ implies $x \in A \cup B$ and $x \in C$ implies $x \in A \cup C$, so again x is in their intersection. Thus any x in $A \cup (B \cap C)$ is also in $(A \cup B) \cap (A \cup C)$, so $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

To see $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ we do the other way around. Start with $x \in (A \cup B) \cap (A \cup C)$. By the definition of intersection, $x \in A \cup B$ and $x \in A \cup C$. We consider two cases: x in A and x not in A. In the first case, $x \in A$, then x is certainly in $A \cup (B \cap C)$ as well. In the second case, where x is not in A, then knowing $x \in A \cup B$ tells us that x is in B and knowing that $x \in A \cup C$ tells us that x is in C, so x is in $B \cap C$. Thus in this case as well we have $x \in A \cup (B \cap C)$.

(4) $A \cup B^c = (A \cap B) \cup (A^c \cap B^c)$
False. Any element in both A and B^c is in the left hand side, but isn't in $A \cap B$ (since it is in B^c and so isn't in B) and isn't in $A^c \cap B^c$ (since it is in A and so not in A^c). For example, if $A = [-1, 2]$ and $B = \mathbb{Z}$ then $\frac{1}{2}$ is in the left side but not the right.

(5) If f and g are surjective then $g \circ f$ is surjective
True. To show $g \circ f$ surjective we need to see that for any $c \in C$ there is $a \in A$ with $g \circ f(a) = c$. Since g is surjective there is a $b \in B$ with $g(b) = c$, and since f is surjective there is $a \in A$ with $f(a) = b$. Then $g \circ f(a) = g(f(a)) = g(b) = c$ as needed.

(6) If $g \circ f$ is surjective then f is surjective
False. For example, if f is the function from $\{UICStudents\} \to \mathbb{N}$ that assigns each student their UIN and $g : \mathbb{N} \to \{even, odd\}$ that assigns each integer to its parity then $g \circ f$ is the map $\{UICStudents\} \to \{even, odd\}$ that assigns each student the parity of their UIN. This is surjective since there are students with even UINs and students with odd UINs, but f is not surjective (for example, there are no students with negative UINs).

(7) If $g \circ f$ is surjective then g is surjective
True. To show that g is surjective we need to show that for any $c \in C$ there is $b \in B$ with $g(b) = c$. We are given that $g \circ f$ is surjective, so there is an $a \in A$ with $g \circ f(a) = c$. Thus $g(f(a)) = c$, so if we take $b = f(a)$ then we have $g(b) = c$ as needed.

(8) If $B \subset Y$ then $f^{-1}(B)^c = f^{-1}(B^c)$
True. An element $x \in f^{-1}(B)^c$ means x is not in $f^{-1}(B)$. By the definition of inverse image, this means $f(x)$ is not in B, which we can write as $f(x) \in B^c$. Since $f(x) \in B^c$ is the definition of $x \in f^{-1}(B^c)$, the two sides mean exactly the same thing.

(9) If $A \subset X$ then $f(A)^c = f(A^c)$
False. An element $y \in Y$ is in the left hand side if y is not in $f(A)$, which means that there is no a in A with $f(a) = y$. The right hand side means that there is an $x \in A^c$ (so an x which is not in A) with $f(x) = y$. These are completely different - one example would be $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$, and $A = [-1, 1]$. Then $f(A) = [0, 1]$, so all the negative numbers are in $f(A)^c$. On the other hand, no negative numbers are in $f(A^c)$ (or in $F(S)$ for any $S \subset \mathbb{R}$) since $f(x) = |x|$ is never negative.

(10) If A_1 and A_2 are subsets of X then $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
True. $y \in f(A_1 \cup A_2)$ means that there is an $x \in A_1 \cup A_2$ with $f(x) = y$. By the definition of union this translates to : there is an $x \in A_1$ with $f(x) = y$ or and $x \in A_2$ with $f(x) = y$. Using again the definition of the image of a set, this becomes $y \in f(A_1)$ or $y \in f(A_2)$. Since this is precisely the definition of $y \in f(A_1) \cup f(A_2)$ the two sets are equal.

(11) If B_1 and B_2 are subsets of Y then $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
True. The definition of inverse image says that $x \in f^{-1}(B_1 \cup B_2)$ means $f(x) \in B_1 \cup B_2$. The definition of union says that this means $f(x) \in B_1$ or $f(x) \in B_2$. Rewriting these using the definition of inverse image, this is the same as $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, which is precisely the definition of $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$.