NIM

1. Rules

NIM is played between two players who take turn making moves. A game position consists of any number of "piles", each of which has some size (a natural number : $1, 2, 3, \ldots$). On their turn a player chooses one pile and reduces it's size by an amount of their choosing, anywhere between reducing the pile size by one to removing the pile entirely. If, on a player's turn there are no piles remaining, so that they have no legal move, that player loses.

A Sample Game:

Our game is being played by Alice and Bob, and starts with three piles, of sizes 1, 2, and 4.

- Alice goes first and reduces the pile of size 4 to size 3. Now there are piles of sizes 1,2, and 3.
- Next, Bob moves by removing the entire pile of size 2, leaving piles of sizes 1 and 3.
- Alice reduces the pile of size 3 to a pile of size 1, leaving two piles both of size 1.
- Bob takes one of the piles, leaving a single pile of size 1.
- Alice takes the last pile, leaving nothing.
- Bob has no move and so loses.

2. Outcomes

3. Copycat

A Story: Suppose Dog and Cat are playing a game of NIM which has only two remaining piles, both of the same size. Dog goes first and makes some move, reducing one of the piles. Cat then just copies Dog, doing the same thing in the second pile. There are now again two piles of equal size, and it is again Dog's move. Cat repeats this copying strategy. At some point Dog will completely remove one of the piles, and then Cat will copy them in removing the other. It will then be Dog's turn with no remaining piles, so Dog will lose.

The moral of our story is that with two equal piles, whoever goes first will lose if their opponent uses this strategy. In general, playing NIM well is about knowing which positions can be won player the player whose turn it is, and which are they will lose (with correct play by their opponent) no matter what they do. Here are some cases we understand already:

(1) If the position has no piles at all, the player to move loses (this is a rule of the game!)
(2) If there is a single pile, the player to move wins

The player to move can simply take the entire pile, leaving the second player in the unfortunate case of the last item.

(3) If there are exactly two piles, and they are of equal size the the player to move loses (see the CopyCat strategy).

(4) If there are exactly two piles, and they are of unequal sizes, then the the player to move wins.

The player to move can take enough items from the larger pile to reduce it to the size of the smaller one, leaving the second player in the unfortunate case of the last item.

What about positions with more than two piles? Some cases we can figure out fairly quickly:

- If there are two piles of equal size and a third pile then the player whose turn it is can take the third pile, leaving their opponent to move in a position we know is a loss for whoever has to move.
- If all the piles are of size one, then the players have essentially no choices in how to move - they just take turns removing piles, so the player to move wins if there are an odd number of piles, and loses if there are an even number.

What about something not covered, like $\{1, 2, 3\}$? We don't see any immediate moves for the first player that turn it into a position on our lists above, so we just need to check all of the legal moves individually to see if they win.

We can eliminate them all fairly quicklu: taking all of any pile leaves two piles of unequal size and we know the opponent has a winning move after that. Likewise, reducing any pile to be equal in size to a smaller pile gives the opponent a winning response (take the entire third pile, leaving two equal ones).

With a bit of thought one can see this covers all the legal moves, so the first player has no winning move, meaning that $\{1, 2, 3\}$ is a loss for the player to move.

3.1. **Formal definitions.** We can formalize the earlier discussion:

- A position P is **type F** if there is a legal move that results in a position of type S.
- A position P is **type S** if there is not a legal move that results in a position of type S (equivalently, if all the legal moves result in a position of type F).

We think of "Type F" as meaning a win for whichever player gets to move first, and type S as meaning it is a win for whichever player gets to move second. The two cases in the definition above then just encapsulate the basic facts that a position is winning for the player to move if they have a move that wins, and is losing for them if there isn't.

It seems like this definition is a bit circular - in order to understand whether a position is type F or type S I need to be able to decide whether some other positions are type F or type S! The reason this doesn't cause any problems is that every time a move is made the position gets smaller - so if you repeatedly apply the definition you eventually get to a position with no piles at all. If you have seen mathematical induction or recursive functions in computer science the you will recognize this kind of process. We will look at it in much more detail as the course goes on.

Notice that for the position with no piles we can use the definition without difficulty : there is no move that results in a position of type S because there is no move at all, so the definition says the position is type S (so a loss for the player to move). So this definition actually includes that rule of the game!

Exercise Try using the definition to work out what the outcome is for a position whose outcome you don't know, like $\{2, 3, 5\}$.