MATH 215, FALL 2018 (WHYTE) SAMPLE MIDTERM

(1) Decide which of the following statements about sets are true and which are false (you do not need to justify your answers):
   (a) $(A \cup B) \cap (C \cup D) = (A \cap C) \cup (B \cap D)$
   (b) $(A \cup B^c) = (A^c \cap B)^c$
   (c) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

Although the problem does not ask for proofs or counter-examples, I will give some here for reference

(a) is False : for example, let $A = D = [0, 1]$ and $B = C = [2, 3]$ then the left hand side is $[0, 1] \cup [2, 3]$ but the right hand side is empty

(b) is True : To prove it, suppose $x \in (A^c \cap B)^c$. By the definition of complemement, this means x is not in $A^c \cap B$. By the definition of intersection, this means x is not in both of A^c or B. In other words, either x is not in A^c or x is not in B. This means exactly that x is in $(A^c)^c \cup B^c$. Since $(A^c)^c = A$, we have $A \cup B^c$ as claimed.

(c) is True : The proof of this was given in class.

(2) For each of the functions below, decide whether they are injective and/or surjective(you do not need to justify your answers):
- $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$
- $g : \mathbb{N} \to \mathbb{N}$ given by $g(n) = n^2$
- $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(n) = 3n - 7$
- $k : \mathbb{R} \to \mathbb{R}$ given by $k(x) = 3x - 7$
- $l : \{cat, ball, dog, door\} \to \{a, b, c, d\}$ defined by sending each word to its first letter

f is both surjective and injective (note that it is important that the domain and range are the sets of non-negative reals, as a map $\mathbb{R} \to \mathbb{R}$ it is neither)

g is injective (since if $n^2 = m^2$ with $n \neq m$ then $n = -m$ but the second is impossible since $\mathbb{N}$ does not contain negative numbers). It is not surjective - for example, there is no natural number n with $n^2 = 2$

h is injective ($3n - 7 = 3m - 7$ can only happen when $n = m$) but not surjective (for example $3n - 7 = 0$ has no solutions in $\mathbb{Z}$)

k is both injective ($3x - 7 = 3y - 7$ can only happen when $x = y$) and surjective (since $3x - 7 = y$ has a solution $x = \frac{y+7}{3}$ which is a real number whenever y is)

l is neither injective (since $l(dog) = l(door) = d$) nor surjective (as none of the words in the domain start with "a").

(3) Let $f : A \to B$ be a function
 (a) Prove that for any set $S \subset B$ we have $f(f^{-1}(S)) \subset S$
 (b) Prove that if f is surjective then for any set $S \subset B$ we have $S \subset f(f^{-1}(S))$
 (c) Given an example to show that (b) is false without the assumption that f is surjective.

Lets start by unpacking what $x \in f(f^{-1}(S))$ means. By the definition of the image of a set, $x \in f(f^{-1}(S))$ means that there is a $y \in f^{-1}(S)$ with $f(y) = x$. Likewise, using the definition of inverse image, $y \in f^{-1}(S)$ means that $f(y) \in S$. Combining these, $x \in f(f^{-1}(S))$ if and only if there is a y with $f(y) = x$ and $f(y) \in S$.

(a) now follows immediately : if $x \in f(f^{-1}(S))$ then we have shown that there is a y with $f(y) = x$ and $f(y) \in S$. But $f(y) = x$ and $f(y) \in S$ certainly implies that $x \in S$.

(b) is similar. If we have $x \in S$ and f surjective then there is a y with $f(y) = x$. Since we assumed we started with $x \in S$ then $f(y) = x$ means $f(y) \in S$ so the characterization of $f(f^{-1}(S))$ we worked out shows that $x \in f(f^{-1}(S))$ so $S \subset f(f^{-1}(S))$.

(c) if f is not surjective then our argument breaks down in (b) because there doesn't need to be a y with $f(y) = x$. Just about any example of such an f will work: for example, if $f(x) = \sin x$ then since $f(x) = 2$ is impossible, we can take any S that contains 2, say $S = [0, \infty)$ and check that $f(f{-}1(S))$ is smaller than S (because 2 is in S and cannot be in the image of f of anything.

(4) Suppose $f : A \to B$ and $g : B \to C$ be functions. Decide whether each of the following is true or false and justify your answer with a proof or counter-example.
   (a) If f and g are injective then $g \circ f$ must be injective
   (b) If f is not injective then $g \circ f$ cannot be injective
   (c) If g is not injective then $g \circ f$ cannot be injective

(a) is True and the proof has been discussed several times in class.

(b) is True: Since f is not injective, there are $a_1 \neq a_2$ so that $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$, or written differently, $g \circ f(a_1) = g \circ f(a_2)$, so $g \circ f$ is not injective.

(c) is False: for example, $g(x) = x^2$ is not injective, but if $f(x) = e^x$ then $g \circ f(x) = g(f(x)) = (e^x)^2 = e^{2x}$ is injective. The point here is that g not being injective means there are $b_1 \neq b_2$ so that $g(b_1) = g(b_2)$, but $g(f(x))$ only sees the elements of B that are in the image $f(A)$, so if this image misses one of the b_1 or b_2 then the composition can still be injective. In the example given this happens because x^2 is not injective only because it ignores signs, but e^x is always positive so there are never any negative numbers that are squared in constructing the composition.