SETS AND FUNCTIONS, MATH 215 FALL 2018 (WHYTE)

1. Intro to Sets

After some work with numbers, we want to talk about **sets**. For our purposes, sets are just collections of objects. These objects can be anything - numbers, words, other sets, etc. We use the notation : $\{\cdots\}$ for a set, with the elements listed in the middle. Some examples:

- $\{1, 2, 3, 4\}$
- $\{cat, dog, house\}$
- $\{2m + 1 : m \in \mathbb{Z}\}$
- $\{0, 1, 2, 3, \ldots\}$

The third and fourth sets listed here show two common methods of defining sets. In the third we are taking every number of the form $2m + 1$ where m is allowed to be any integer. For the fourth we have used $\ldots$ to indicated that the sequence continues in the given pattern indefinitely. This is not completely rigorous, since we have not specified what the pattern is - this notation should not be used if there is any doubt about what is intended.

Definition 1.1. *Two sets S and T are equal, written $S = T$, if they have the same elements. In other words, if $x \in S \iff x \in T$.*

Definition 1.2. *Given sets S and T , we say S is a subset of T, written $S \subset T$, if all the elements of S are also elements of T (so $x \in S \implies x \in T$).*

The following is a common strategy for proving two sets are the same:

Proposition 1.3. *Given sets S and T, $S = T$ if and only if both $S \subset T$ and $T \subset S$.*

We can build new sets out of ones we already have.

Definition 1.4. *Given a set S, its complement, written S^c is defined by*

$$x \in S^c \iff x \notin S$$

Definition 1.5. *Given sets S and T, their union, $S \cup T$ is defined by*

$$x \in S \cup T \iff (x \in S \text{ or } x \in T)$$

Definition 1.6. *Given sets S and T, their intersection, $S \cap T$ is defined by*

$$x \in S \cap T \iff (x \in S \text{ and } x \in T)$$

One common set is the set which has no elements, called the **empty set**. We will use the symbol $\emptyset$ for this set.

Some basic facts about these operations on sets:

Proposition 1.7. *Let A,B, and C be sets.*

- $A \cup B = B \cup A$
- $(A \cup B)^c = (A^c \cap B^c)$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $A \cap B = B \cap A$
- $(A \cap B)^c = (A^c \cup B^c)$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
- $A \cap \emptyset = \emptyset$
- $A \cup \emptyset = A$
- $\emptyset \subset A$

(hint :All of these can be proven in a straightforward way from the definitions, sometimes with the help of Prop 1.3).

Sets can have other sets as elements, and this can be confusing. For example:

Question 1.8. *Let* $S = \{a, b, \{x, y\}\}$

- *How many elements does S have?*
- *Is x an element of S?*
- *Is a an element of S?*
- *Is $\{x, y\}$ an element of S?*
- *Is $\{x, y\}$ a subset of S?*
- *Is $\{a, b\}$ an element of S?*
- *Is $\{a, b\}$ a subset of S?*

One source of examples of sets with other sets as elements is the **power set**:

Definition 1.9. *Let S be a set. The* **power set** *of S, written $P(S)$, is the set of all subsets of S. So $x \in P(S) \iff x \subset S$.*

Example 1.10. *If $S = \{a, b\}$ then $P(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$*

Proposition 1.11. *For any set S, $P(S)$ has at least one element.*

Proposition 1.12. *If S is a set such that $P(S)$ has only one element then $S = \emptyset$.*

Work out examples of the power sets of some (small) sets. In particular, for the example above of $S = \{a, b\}$, what is $P(P(S))$?

Problem 1.13. *Suppose S is a set with n elements. Make a conjecture about how many elements $P(S)$ has, then prove your conjecture using induction (hint: the answer is also an explanation for the name "power set").*

2. Functions

Informally, a function from a set A to a set B is a rule that takes as input an element of A and gives, as output, an element of B. We will call A the **domain** of the function and B the **range**. You have likely already encountered many functions. Some examples:

- The function that squares a real number. In this case the input and output are both real numbers, so $A = B = \mathbb{R}$. In this case you are probably used to seeing the function given a name, like f, and the rule written by an equation like $f(x) = x^2$.
- The function that assigns to each student their UIN. In this case the input is a student (at the U of I) and the output is a number. So we could take A as the set of students at U of I and B is the set of numbers. We know that UINs are all 9-digits, so we could also write this as a function where B is the set of 9-digit numbers. We cannot let A be the set of all students because not every student has a UIN (like kindergarten students or students at Northwestern, etc) although we could restrict ourselves to students at UIC. It is important to remember that we include the sets A and B as part of the definition of a function, so all of these are different (but related) functions from our perspective.
- The function that assigns to each word its first letter. Here A is the set of words and B is the set of letters. To be precise and fully define this function I would need to specify what counts as a word and what counts as a letter.
- The function that assigns to a NIM position the set of positions that can result from a legal move. Here A is the set of NIM positions but B is the set of sets of NIM positions, or more formally $P(A)$.
- The function that assigns to a quadratic equation its solutions. We can write a quadratic equation as

  $$ax^2 + bx + c$$

  where a, b, c are real numbers, so we can encode the input set A as $\mathbb{R}^3$. You have probably memorized a formula for this function :

  $$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  This hides a number of subtleties: is it really defined if $a = 0$? What about if $b^2 - 4ac < 0$? Even in "normal" cases it is really two numbers (see the $\pm$) not just one. So A probably needs to be only those elements of $\mathbb{R}^3$ with $a \neq 0$, and B needs to be $P(\mathbb{R})$ rather than just $\mathbb{R}$ as the output is a set of real numbers (in this case containing 0, 1, or 2 elements).

- The function that assigns a set of integers the sum of its elements. This only makes sense if it is a finite set, so A is all finite subsets of $\mathbb{Z}$, and the output is in $\mathbb{Z}$.

Examples like the last two where the domain and/or range is a set of sets are quite common both in mathematics and in most fields of application.

It is often the case that we would like to reverse a function: to go backwards from a UIN to a student or from a real number to its square root. If we think about these examples we see there are some problems: not every 9-digit number is someone's UIN, so this "lookup by UIN" function is not actually defined on the set of all 9-digit numbers. This also happens for real numbers, where negative numbers don't have square roots. The square root function has another problem as well, that most numbers actually have two square roots. These two issues are quite common, and they have names:

Definition

- A function $f : A \to B$ is **surjective** if for every $b \in B$ there is some $a \in A$ with $f(a) = b$
- A function $f : A \to B$ is **injective** if for any $a_1 \neq a_2$ in A we have $f(a_1) \neq f(a_2)$
- A function $f : A \to B$ if **bijective** if it is both injective and surjective.

Exercise Go back to the list of example functions above and decide if each is injective/ surjective/ bijective.

One basic way of building new functions out of old ones is composition, which just means applying one function and the applying another to the output. Formally:

Definition If f is a function from A to B and g is a function from B to C, then the composition $g \circ f$ is defined by

$$g \circ f(a) = g(f(a))$$

Notice that we need to have to range of f match up with the domain of g, otherwise it doesn't make sense to apply g to the output of f.

Example If f:US Taxpayers $\to$ 9-digit numbers is the function that assigns to each taxpayer their social security number and g is the function from 9-digit numbers to 4-digit numbers that cuts each number down to its last four digits then the composition $g \circ f$ is the function that assigns to each taxpayers the last four digits of their SSN.

Exercise Let $f : A \to B$ and $g : B \to C$ be functions. Which of the following are true?

(1) If f is injective and g is injective then $g \circ f$ is injective.

(2) If f is injective and g is surjective then $g \circ f$ is injective.
(3) If f is surjective and g is injective then $g \circ f$ is injective.
(4) If f is surjective and g is surjective then $g \circ f$ is injective.
(5) If f is injective and g is injective then $g \circ f$ is surjective.
(6) If f is injective and g is surjective then $g \circ f$ is surjective.
(7) If f is surjective and g is injective then $g \circ f$ is surjective.
(8) If f is surjective and g is surjective then $g \circ f$ is surjective.

We can formalize what we mean by "reversing" a function:

Definition Functions $f : A \to B$ and $g : B \to A$ are **inverse functions** if

- For all $a \in A$ we have $g(f(a)) = a$ and
- For all $b \in B$ we have $f(g(b)) = b$

Notice that these conditions are both statements about the compositions $f \circ g$ and $g \circ f$, namely that $f \circ g(b) = b$ for all $b \in B$ and $g \circ f(a) = a$ for all $a \in A$. This point of view is useful enough that it has its own definition: for any set S the **identity function on** S is the function $I_S : S \to S$ defined by $I_S(s) = s$ for all $s \in S$. In this language the two conditions become $f \circ g = I_B$ and $g \circ f = I_A$.

Here are some statements about the relationship between these concepts for you to prove, and some related questions for you to think about:

(1) Let $f : A \to B$ be a function. There is a function $g : B \to A$ with $f \circ g = I_B$ if and only if f is injective. Can there be more than one such function g?

(2) Let $f : A \to B$ be a function. There is a function $g : B \to A$ with $g \circ f = I_A$ if and only if f is surjective. Can there be more than one such function g?

(3) Let $f : A \to B$ be a function. There is a function $g : B \to A$ which is inverse to f if and only if f is bijective. Can there be more than one such function g?

2.1. **Images and Inverse Images.** When studying a function $f : A \to B$, we often care not just about what happens for individual elements of A but also about what happens for sets of elements. We will discuss two common constructions, one that turns a subset of A into a subset of B and another that turns a subset of B into a subset of A.

The first is more straight forward: given a subset $S \subset A$ we define $f(S)$ **the image of** S by : $b \in f(S)$ if and only if there is an $s \in S$ such that $b = f(s)$. In less formal language, the image of S is the sets of elements of B that you get by plugging in elements of S. So images of sets answer the question : which elements of the range come from a restricted part of the domain?

As an example, if $g : \{ \text{ 215 Students } \} \to \{A, B, C, D, F\}$ is the function that assigns every student their course grade, then we might want to ask about the grades particular kinds of students (the subset of graduating seniors, the subset of math majors, the subset of students who live on campus, etc).

The inverse image construction answers questions from the opposite perspective - if we look at a restricted set of grades, which students got them? So, for example, which students pass? Formally:

If $f : A \to B$ is a function and $S \subset B$ then we define $f^{-1}(S) \subset A$, the **inverse image** of S by : $a \in f^{-1}(S)$ if and only if $f(a) \in S$.

Warning: The notation for inverse image is a potential source of confusion. It looks like it is just the image but using the inverse function. However, the inverse images of sets are defined for all functions, not only ones with inverses and it is easy to lose track of this and accidentally assume the function is invertible and so draw false conclusions.

These constructions are related to some of our earlier definitions, for example:

Proposition 2.1. *Let $f : A \to B$ be a function. The following three conditions are equivalent:*

(1) *f is surjective*
(2) *$f(A) = B$*
(3) *$f^{-1}(S)$ is empty only if S is empty*

Exercise Prove the proposition. Can you come up with interpretations of injectivity using image and inverse images?

Questions Here are some questions about how images and inverse images work for a function $f : A \to B$. Decide whether each statement one is true or false, and either prove it or give a counter-example:

(1) $f(f^{-1}(S)) = S$ for all $S \subset B$
(2) $f^{-1}(f(S)) = S$ for all $S \subset A$
(3) $f(S \cup T) = f(S) \cup f(T)$ for all S and T subsets of A
(4) $f(S \cap T) = f(S) \cap f(T)$ for all S and T subsets of A
(5) If $S \subset T \subset A$ then $f(S) \subset f(T)$
(6) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$ for all S and T subsets of B
(7) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$ for all S and T subsets of B
(8) If $S \subset T \subset B$ then $f^{-1}(S) \subset f^{-1}(T)$

Do any of the ones that are false become true if we add an assumption that f is injective? that f is surjective? bijective?