You may refer to the textbook and your notes, but not any other resources/help. Please email your exam (as a pdf) to kwhyte@uic.edu by 5pm on Friday May 8th.

1. Let $G$ be a group of order 2020, and assume $G$ has an element $a$ of order 101 and an element $b$ of order 20.
   (a) Show that $\langle a \rangle \langle b \rangle = G$
   (b) Show $\langle a \rangle$ is a normal subgroup
   (c) Show that $ba = a^k b$ for some $k$. Which $k$ are possible?
   (d) Determine how many possibilities there are for the group $G$ (counting any two isomorphic groups as the same).

2. Find all the ring homomorphisms from $\mathbb{Z}[x]/<x^2 + 5>$ to $\mathbb{Z}/12\mathbb{Z}$. Which of them send 1 to 1? For each homomorphism, determine its kernel and image.

3. (a) Is every ideal in $\mathbb{Z}/2\mathbb{Z}[x]$ principle? Prove your answer is correct.
   (b) Is every ideal in $\mathbb{Z}/4\mathbb{Z}[x]$ principle? Prove your answer is correct.
   (c) Is every ideal in $\mathbb{Z}/6\mathbb{Z}[x]$ principle? Prove your answer is correct.

4. (a) Show that for any irreducible quadratic polynomial $P$ in $\mathbb{Q}[x]$ there is a non-zero integer $d$ so that $\mathbb{Q}[x]/<P>$ is isomorphic to $\mathbb{Q}[\sqrt{d}]$.
   (b) Show that if $d_1$ and $d_2$ are non-zero integers and $\mathbb{Q}[\sqrt{d_1}]$ is isomorphic to $\mathbb{Q}[\sqrt{d_2}]$ then $d_2 = r^2 d_1$ for some $r \in \mathbb{Q}$.

5. Let $R$ be a commutative ring. A linear map $R \rightarrow R$ is a function $f$ of the form $f(x) = ax + b$ for some $a$ and $b$ in $R$.
   (a) Show that a linear map $f(x) = ax + b$ is an invertible function from $R$ to $R$ if and only if $a$ is a unit. When $a$ is a unit the map $f$ is called a similarity, and when $a = 1$ the map is called a translation.
   (b) Show that the set of similarities of $R$ is a group under composition, that the set of translations is a normal subgroup, and that the quotient group is isomorphic to the group of units of $R$.
   (c) For which $n$ is the group of similarities of $\mathbb{Z}/n\mathbb{Z}$ an Abelian group?
   (d) What is the center of the group of similarities of $\mathbb{Z}/n\mathbb{Z}$?