MATH 330, SPRING 2020, FINAL EXAM

You may refer to the textbook and your notes, but not any other resources/help. Please email your exam (as a pdf) to kwhyte@uic.edu by 5pm on Friday May 8th.

(1) Let G be a group of order 2020, and assume G has an element a of order 101 and an element b of order 20.
  (a) Show that $< a >< b >= G$
  (b) Show $< a >$ is a normal subgroup
  (c) Show that $ba = a^k b$ for some k. Which k are possible?
  (d) Determine how many possibilities there are for the group G (counting any two isomorphic groups as the same).

(2) Find all the ring homomorphisms from $\mathbb{Z}[x]/ < x^2 + 5 >$ to $\mathbb{Z}/12\mathbb{Z}$. Which of them send 1 to 1? For each homomorphism, determine its kernel and image.

(3) (a) Is every ideal in $\mathbb{Z}/2\mathbb{Z}[x]$ principle? Prove your answer is correct.
  (b) Is every ideal in $\mathbb{Z}/4\mathbb{Z}[x]$ principle? Prove your answer is correct.
  (c) Is every ideal in $\mathbb{Z}/6\mathbb{Z}[x]$ principle? Prove your answer is correct.

(4) (a) Show that for any irreducible quadratic polynomial P in $\mathbb{Q}[x]$ there is a non-zero integer d so that $\mathbb{Q}[x]/ < P >$ is isomorphic to $\mathbb{Q}[\sqrt{d}]$.
  (b) Show that if d_1 and d_2 are non-zero integers and $\mathbb{Q}[\sqrt{d_1}]$ is isomorphic to $\mathbb{Q}[\sqrt{d_2}]$ then $d_2 = r^2 d_1$ for some $r \in \mathbb{Q}$.

(5) Let R be a commutative ring. A linear map $R \to R$ is a function f of the form $f(x) = ax + b$ for some a and b in R.
  (a) Show that a linear map $f(x) = ax + b$ is an invertible function from R to R if and only if a is a unit. When a is a unit the map f is called a similarity, and when $a = 1$ the map is called a translation.
  (b) Show that the set of similarities of R is a group under composition, that the set of translations is a normal subgroup, and that the quotient group is isomorphic to the group of units of R.
  (c) For which n is the group of similarities of $\mathbb{Z}/n\mathbb{Z}$ an Abelian group?
  (d) What is the center of the group of similarities of $\mathbb{Z}/n\mathbb{Z}$?