MATH 445, PROBLEM SET # 2

(1) Let $X$ be a Hausdorff topological space. Let $K(X)$ be the set of non-empty compact subsets of $X$. We define a topology on $K(X)$ as follows and study its properties.

(a) Given a finite set $F = \{U_1, \ldots, U_n\}$ of open sets in $X$ define a set $V_F$ in $K(X)$ whose elements are the compact subsets of $X$ contained in the union of the $U_i$ and which intersect every $U_i$ in a non-empty set. Prove that the collection of $V_F$ for all possible $F$ are a basis for some topology on $K(X)$. For the remainder of this problem let $K(X)$ be given this topology.

(b) Prove that $K(X)$ is Hausdorff.

(c) Prove that for any $n \in \mathbb{N}$ the map $X^n \to K(X)$ sending $(x_1, \ldots, x_n)$ to the set $\{x_1, \ldots, x_n\}$ is continuous.

(d) Prove the the subset of $K(X)$ consisting of all finite subsets of $X$ is dense in $K(X)$.

(e) If $A \subset X$ let $K(A) \subset K(X)$ be those compact sets in $X$ which are subsets of $A$. Prove that $K(A)$ is open in $K(X)$ if $A \subset X$ is open and closed in $K(X)$ if $A$ is closed in $X$.

(f) Show that $K(X)$ is connected iff $X$ is connected.

(g) Show that if $X$ is non-compact then $K(X)$ is non-compact.

(h) (harder) Is it true that $X$ compact implies $K(X)$ compact?

(2) Show that if $X$ and $Y$ are compact and Hausdorff and $f : X \to Y$ is a continuous bijection then $f$ is a homeomorphism. Give examples to show you cannot omit either hypothesis.

(3) For any space $X$ we define its one point compactification, $X^+$, as the set $X \cup \{\infty\}$ with the topology whose open sets are of two kinds: open subsets of $X$ and subsets $U \cup \{\infty\}$ where $U \subset X$ is open with $X \setminus U$ compact.

(a) Show that this does define a topological space and that it is compact.

(b) Show that the inclusion map of $X$ into $X^+$ is continuous and that the image is dense if $X$ is non-compact.

(c) If $X$ is Hausdorff is $X^+$ Hausdorff? If not, what extra assumptions are needed?

(4) Show that the Cantor set ( $\{0,1\}^\mathbb{N}$ with the product topology) is compact.