NIL

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Let M be $\mathbb{R}^3$ equipped with the metric in which the following vector fields:

$$U = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$$

$$V = \frac{\partial}{\partial y}$$

and

$$W = \frac{\partial}{\partial z}$$

are an orthonormal basis.

1. Preliminaries

Since we're given the metric via an orthonormal basis of vector fields which are not coordinates, we need to compute the Lie brackets. This gives:

$$[U, W] = [V, W] = 0$$

and

$$[U, V] = -W$$

It will also be convenient to sometimes switch the basis os vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$. The transformation to these from U,V, and W is:

$$\frac{\partial}{\partial x} = U - yW$$

$$\frac{\partial}{\partial y} = V$$

$$\frac{\partial}{\partial z} = W$$

2. The Levi-Civita Connection

Our general formula for the Levi-Civita connection says:

$$2 < D_A B, C >= A < B, C > +B < A, C > -C < A, B >$$
$$+ < [A, B], C > - < [A, C], B > - < [B, C], A >$$

If the vector fields are part of an orthonormal basis, then all their inner products are constant (and equal to 0 or 1) and so the first three terms are zero. The only non-zero Lie bracket among U, V, and W is $[U, V] = -W$, so the only way to get any non-zero terms with A, B, and C coming from U, V, and W is to have each occur once, in which case $< [U, V], W >$ occurs as exactly one of the terms, possibly with a minus sign. In other words:

$$< D_U V, W >= -\frac{1}{2}$$
$$< D_V U, W >= \frac{1}{2}$$
$$< D_U W, V >= \frac{1}{2}$$
$$< D_W U, V >= \frac{1}{2}$$
$$< D_V W, U >= -\frac{1}{2}$$
$$< D_W V, U >= -\frac{1}{2}$$

and all other such inner product terms are zero. Thus, since U, V, W is an orthonormal basis, we get:

$$D_U V = -\frac{1}{2} W$$
$$D_V U = \frac{1}{2} W$$
$$D_U W = D_W U = \frac{1}{2} V$$
$$D_V W = D_W V = -\frac{1}{2} U$$
$$D_U U = D_V V = D_W W = 0$$

This determines the connection via linearity and the Leibnitz rule. In particular, we can get the connection in coordinates from this easily. For example:

$$D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = D_{U-yW}V = D_U V - yD_W V$$
$$= -\frac{1}{2}W + \frac{1}{2}yU = -\frac{1}{2}\frac{\partial}{\partial z} + \frac{1}{2}y(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z})$$
$$= \frac{1}{2}(y\frac{\partial}{\partial x} + (-1+y^2)\frac{\partial}{\partial z})$$

similarly, we compute:

$$D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial y}$$
$$D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = -y\frac{\partial}{\partial y}$$
$$D_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = D_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z} = 0$$
$$D_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z} = -\frac{1}{2}(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z})$$

3. Geodesics

We will try to understand a geodesic $\sigma(t) = (x(t), y(t), z(t))$. We have two choices: we can compute in coordinates, or convert to the orthonormal basis.

In coordinates, we write $c'(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y} + z'(t)\frac{\partial}{\partial z}$. As usual, when we expand the geodesic equation $D_{c'}c' = 0$ in coordinates, we get:

$$x''\frac{\partial}{\partial x} + y''\frac{\partial}{\partial y} + z''\frac{\partial}{\partial z} + x'^2 D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} + y'^2 D_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} + z'^2 D_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z}$$
$$+2x'y'D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} + 2x'z'D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z} + 2y'z'D_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z} = 0$$

Substituting the calculations of the connection in coordinates from the previous section gives:

$$(x''+x'y'y-y'z')\frac{\partial}{\partial x}+(y''+x'z'-yx'^2)\frac{\partial}{\partial y}+(z''+x'y'(-1+y^2)-y'z'y)\frac{\partial}{\partial z} = 0$$

In the alternative method, we start at the same place, $c' = x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}$ and convert to U, V, and W to get $c' = x'U + y'V - (z' - yx')W$. Then, the geodesic equation $D_{c'}c' = 0$ expands to:

$$D_{c'}x'U + D_{c'}y'V + D_{c'}(z' - yx')W = 0$$

expanding again:

$$x''U + x'D_{c'}U + y''V + y'D_{c'}V + (z'' - y'x' - yx'')W + (z' - yx')D_{c'}W = 0$$

and finally

$$(x'' - y'(z' - x'y))U + (y'' + x'(z' - x'y))V + (z'' - yx' - yx'')W = 0$$

If we substitute the expression for U, V and W into this, we see we have exactly the same system of equations just written differently. We'll work with the second one as it is slightly easier. This means we're looking at the system of equations:

$$\begin{aligned} x'' - y'(z' - x'y) &= 0 \\ y'' + x'(z' - x'y) &= 0 \\ z'' - yx' - yx'' &= 0 \end{aligned}$$

The first thing to notice is that the expression $z' - x'y$ shows up in several places. The third equation simply says that this expression is constant. So, name this constant K. Our new system is:

$$\begin{aligned} x'' - Ky' &= 0 \\ y'' + Kx' &= 0 \\ z' - x'y &= K \end{aligned}$$

Looking only at the first two equations, we see that the projection of the geodesic to the $x - y$ plane has its acceleration vector perpendicular to its velocity vector and K times as long. This characterizes circular motion around at an angular speed of K. When $K = 0$ we can see the equations give a straight line in the $x - y$ plane, which can be thought of as a limiting case.

We can write this as :

$$\begin{aligned} x &= A + B\sin(Kt + C) \\ y &= D + B\cos(Kt + C) \end{aligned}$$

Our equation for z' then gives:

$$z' = K + BDK\cos(Kt + C) + B^2K\cos^2(Kt + C)$$

which we can solve to get:

$$z = E + K(1 + \frac{1}{2}B^2)t + \frac{1}{2}B^2\sin(Kt + C)\cos(Kt + C)$$

This geodesic begins at $(A+B\sin C, D+B\cos C, E+\frac{1}{2}B^2\sin C\cos C)$. Since Nil is homogeneous, it is enough to understand geodesics starting at $(0,0,0)$. In which case the equations simplify to:

$$x = B(\sin(Kt+C) - \sin C)$$
$$y = B(\cos(Kt+C) - \cos C)$$
$$z = (1+\frac{1}{2}B^2)Kt + \frac{1}{2}B^2(\sin(Kt+C)\cos(Kt+C) - \sin C\cos C)$$

Notice that the speed of the geodesic at $t=0$ is $x'(0)^2+y'(0)^2+z'(0)^2$, and that since $z' = K+x'y$ and $y(0)=0$, we know that $z'(0)=K$. This makes it easy to compute that the speed of the geodesic given by these equations is just $|K|\sqrt{B^2+1}$. Also note that at times $t=\frac{2\pi n}{K}$ for $n\in\mathbb{Z}$ all the trig terms drop out, and we have just $c(t) = (0,0,(1+\frac{1}{2}B^2)2\pi n)$. In this way we can find infinitely many different geodesics between $(0,0,0)$ and any fixed $(0,0,z)$.

Thus Nil is an example where the exponential map $T_{(0,0,0)}Nil \to Nil$ is not a diffeomorphism, even though Nil is diffeomorphic to $\mathbb{R}^3$. Which is shortest? If we travel the geodesic until $t=\frac{2\pi n}{K}$ then the length is $\frac{2\pi|n|}{|K|}$ times the speed of the geodesic, which is $|K|\sqrt{B^2+1}$. Thus the length of this geodesic between $(0,0,0)$ and $(0,0,z)$ is $2\pi|n|\sqrt{B^2+1}$. On the other hand, to end up at $(0,0,z)$ we must have $2\pi n(1+\frac{1}{2}B^2) = z$, so $B^2 = \frac{z}{\pi n} - 2$. This is only possible if z and n have the same sign (assume it is positive, the other case is identical by symmetry), and if $n \leq fracz2\pi$. The "obvious" geodesic $c(t) = (0,0,t)$ between $t=0$ and $t=z$ has length $|z|$. The all implies that the above geodesics are never minimizing to $(0,0,z)$ if $|n|>1$, the obvious geodesic is minimizing only until $z=2\pi$ after which the geodesics above with $n=1$ are minimizing. This gives that $d((0,0,0),(0,0,z))$ is $|z|$ if $|z|\leq 2\pi$, and $d((0,0,0),(0,0,z))$ is $2\sqrt{\pi}\sqrt{|z|-\pi}$ if $|z|\geq 2\pi$. Notice that for z large this says the distance is about the square root of the Euclidean distance. Can you find the distance from $(0,0,0)$ to a general point (x,y,z)? What does the cut locus look like in Nil?

Let's look at the geodesics again, and try to understand them more geometrically. We know that the projection to the $x-y$ plane is moving in a circle, at constant speed. The difficulty is z, which we obtained from $z'-x'y=K$. If we write the as $z'=K+x'y$ we see that we can find the change in z coordinate by integrating $K+ydx$ along the curve (a circular arc) in the $x-y$ plane. The K part is easy, it just contributes a constant Kt to the integral. To integrate the form ydx along the curve, we exploit Stokes' theorem and the fact that $d(ydx) = -dxdy$.

Suppose our circular arc runs from $(0,0)$ to (x,y), and suppose the center of the circle is at some point (a,b). Consider the "pie wedge" consisting of the radial lines from (a,b) to $(0,0)$ and (x,y). The integral of ydx over this wedge is the area enclosed. Since it is a circle of radius B and the arc has angular speed K, this area is $\frac{1}{2}B^2Kt$. The integral along the ray from (a,b) to $(0,0)$ is a constant independent of t. The integral along the line is more complicated to calculate. We can see clearly from this, without calculating, that z will grow like $(1+\frac{1}{2}B^2)Kt$ plus a term which is periodic with a period of $\frac{2\pi}{K}$.

4. Curvature

We start by calculating the curvature tensor R. Recall that:

$$R(A,B,C,D) =< D_A D_B C - D_B D_A C - D_{[A,B]}C, D >$$

Using the calculation of the connection:

$$R(U,V,U,V) = \frac{3}{4}$$
$$R(U,V,U,W) = 0$$
$$R(U,V,V,W) = 0$$
$$R(U,W,U,W) = -\frac{1}{4}$$
$$R(U,W,V,W) = 0$$
$$R(V,W,V,W) = -\frac{1}{4}$$

The values of R of all the rest of the combinations of U, V, and W are determined by these using the symmetry properties of R.

Now, suppose that have a 2-plane, P, in $T_(0,0,0)Nil$ which is given by an equation $aU + bV + cW = 0$. The vectors $X = -bU + aV$ and $Y = -cU + aW$ give a basis for P unless $a = 0$. We can expand using linearity:

$$R(X,Y,Y,X) = -bR(U,Y,Y,X)+aR(V,Y,Y,X) = -b(-cR(U,U,Y,X)+aR(U,W,Y,X))+\dots$$

$$= a^2(a^2R(V,W,W,V) + b^2R(U,W,W,U) + c^2R(V,U,U,V)$$
$$+2bcR(U,V,U,W) - 2acR(U,V,V,W) + 2abR(U,W,V,W))$$

which evaluates to $\frac{1}{4}a^2(a^2 + b^2 - 3c^2)$. The sectional curvature of P is given by:

$$\kappa(P) = \frac{R(X,Y,Y,X)}{||X||^2||Y||^2- < X,Y >^2}$$

We have $||X||^2 = a^2 + b^2$, $||Y||^2 = a^2 + c^2$, and $< X, Y >= bc$, so the denominator evaluates to $a^2(a^2 + b^2 + c^2)$. Putting it all together gives:

$$\kappa(P) = \frac{a^2 + b^2 - 3c^2}{4(a^2 + b^2 + c^2)}$$

This formula extend to the case $a = 0$ by continuity. If we normalize our choice of equation for P so that $a^2 + b^2 + c^2 = 1$ (in other words $aU + bV + cW$ is a unit vector perpendicular to P) then we have that $\kappa = \frac{1}{4} - c^2$. Let P_0 be the plane $a = b = 0$ (the $x - y$ plane). For any P we have that $c = \cos\theta$ where θ is the angle between P and P_0. So we get:

$$\kappa(P) = \frac{1}{4} - \cos^2\theta$$

In particular we see $\kappa(P) = 0$ if $\theta = \frac{\pi}{3}$, and κ is negative for planes closer to P_0 and positive for planes farther. The minimum curvature occurs only for P_0 and is $-\frac{3}{4}$ and the maximal curvature is $\frac{1}{4}$ which occurs for any plane meeting P_0 at a right angle, or in other words, and plane containing the z-axis.

A similar picture holds at al points of Nil by homogeneity. The maximal curvatures always occur for planes containing the z-axis, but the unique plane of minimal curvature is not always the $x - y$ plane (it is the UV plane, which changes in coordinates as (x, y, z) varies).