Index theory with bounded geometry, the uniformly finite $\hat{A}$ class, and infinite connected sums

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Introduction

Our primary focus is the category of manifolds of bounded geometry, meaning that one has bounds on the curvature tensor, its derivatives, and on the injectivity radius. We think of these manifolds as having a chosen bounded distortion class of metrics, two metrics being equivalent if the norm of the identity map (in both directions) is bounded. Unless otherwise stated, all manifolds in the paper are assumed of this type.

These definitions are designed to reflect the restrictions imposed on a non-compact manifold which is controlled in some way by a compact manifold. The most common example of this is a covering of a compact manifold. Any metric on the base gives a metric of bounded geometry on the cover, and any two such metrics are bounded distortion equivalent. Similarly, leaves of foliations of compact manifolds have a canonical bounded distortion class of metric of bounded geometry.

We try to understand index theory for these manfolds, and in particular, questions of positive scalar curvature. Generally, the appropriate notion here is uniformly positive scalar curvature, meaning the scalar curvature is bounded away from zero from below. When we say that M admits a metric of positive scalar curvature, we mean that within the chosen bounded distortion class of metrics there is a metric of uniformly positive scalar curvature.

To understand positive scalar curvature we need an appropriate generalization of the $\hat{A}$ class. One interesting feature which emerges is that this class lives in a non-Hausdorff homology group, and thus standard C^* algebra methods do not apply. It turns out that to understand this class requires rather delicate spectral estimates for Dirac operators on the boundaries of certain compact submanifolds. The resulting theorems have unexpected applications to compact manifolds.

The motivation for this work comes from some interesting infinite connected sum examples studied in [BW1] and [Ro]. As these examples provide good motivation for the definition of the $\hat{A}$ class we use, and are interesting on their own, we discuss them in some detail in the first section. The gap between the obstruction of [Ro] and the construction in [BW2] is a direct consequence of the non-Haudsorff problem mentioned above. Our theorems solve this problem, and provide a complete picture of these examples.

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1 Infinite Connected Sums

Let M be a manifold of bounded geometry and S a discrete subset of M. Given N, a compact manifold, we form a new manifold, $M\#_S N$, by connected summing a copy of N at each point of S. For this manifold to have a well defined bounded distortion class of metrics of bounded geometry one needs that points of S are uniformly separated, meaning that for some $\varepsilon > 0$ any two distinct points of S are at least ε apart. Such a set is called *uniformly discrete*. One important example of this construction is for M the universal cover of some X and S an orbit of $\pi_1(X)$. Then $M\#_S N$ is just a cover of $X\#N$.

Now suppose M has positive scalar curvature. We wish to understand when $M\#_S N$ admits a metric of positive scalar curvature. We restrict our attention to the case of M and N spin, and N simply connected. If we are in the covering space situation, there is a positive scalar curvature metric invariant under the covering group iff $\hat{A}(N) = 0$. We are looking to generalize the $\hat{A}$ class as an obstruction to positive scalar curvature, so we will typically assume $\hat{A}(N) \neq 0$.

A natural first question is whether it is ever possible, keeping bounded geometry, to have a metric of positive scalar curvature in the presence of any such "obstructing" N. Examples showing this is possible are constructed in [BW2]. We sketch their construction below as it gives insight about the kind of obstructions that arise in our main theorem, Theorem 2.1.

Lemma 1.1. *([BW2]) Let M be non-compact, spin, and of uniformly positive scalar curvature, and N be compact, spin, and simply connected. The manifold $M\#N$ admits a metric of uniformly positive scalar curvature.*

Proof. Choose a geodesic ray p in M. Let $\hat{M}$ be the infinite connected sum of M with copies of N at $p(2i)$ and copies of $\bar{N}$ (N with opposite orientation) at $p(2i+1)$, for every i. We view $\hat{M}$ in two ways. First, we think of the copies of N as coming in pairs, at $p(2i)$ and $p(2i+1)$. Each of these pairs is a copy of $N\#\bar{N}$. This manifold is null cobordant, indeed it is the boundary of $N \times [0,1]$. Further, as N is simply-connected, the null cobordism can be realized by a sequence of surgeries of index greater than 1.

The collection of all these surgeries is unfiormly locally finite, and thus can be carried out simultaneously without leaving the bounded geometry category. Using surgery in positive scalar curvature (see [GL] and [SY]), we can also carry out the surgeries keeping positive scalar curvature. Note that to keep the metric in bounded distortion class of the connected sum, one changes the metric only in a neighborhood of the sphere one is surgering. The resulting distortion in that neighborhood depends on the lower bound for the scalar curvature in M, and thus the assuption of uniformly positive scalar curvature cannot be weakened to merely positive scalar curvature.

Now view $\hat{M}$ similarly, but with the copies of N paired as $p(2i-1)$ and $p(2i)$. One can similarly carry out surgeries here, leaving just a single copy of N at $p(0)$. Thus, since $\hat{M}$ has positive scalar curvature so does $M\#N$.

□

Notice that we did not really need that p was a geodesic, just that the time it spends in any ball is uniformly bounded in the radius. Thus to carry out the above construction for $M\#_S N$, one needs such tails from each point of S which spread out in such a way that only a uniformly bounded number pass through any ball. The existence of such "tails" is a homological question.

We say that c, an i-chain in M, is uniformly finite if there is a bound on the diameter of simplices in the support of c and for every r there is an upper bound C_r on the sum of the absolute values of the coefficients of the simplices which intersect any r-ball. We denote these chains by $C_i^{uf}(M)$, and the corresponding homologies by $H_i^{uf}(M)$.

Any S as above gives a natural element of $C_0^{uf}(M)$, and a collection of tails gives a null homology of that class. We denote that class corresponing to S in $H_0^{uf}(M)$ by $[S]$. We have sketched the proof of:

Theorem 1.2. *([BW2]) Let M, N, and S be as above. If $[S] = 0$ then $M\#_S N$ admits a metric of positive scalar curvature.*

To get a feel for the nature of the obstruction measured by H_0^{uf}, consider the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. For any r, the number of lattice points in a ball of radius r is about r^2, while the perimeter is about r. Thus, as r increases, the number of tails crossing the boundary in some bounded region must increase unboundedly. This shows that $[\mathbb{Z}^2]$ is non-zero.

This is essentially the only obstruction to finding tails.

Theorem 1.3. *([BW1], [Wh]) If $c \in C_0^{uf}(M)$ then $[c] = 0$ iff there are r and C such that for any $R \subset M$ one has:*

$$|\Sigma_{\sigma \in R} c_\sigma| \leq C vol(\partial_r R)$$

Recall that a *regular sequence* is a sequence R_i of subsets of M for which for any r,

$$\lim_{i \to \infty} \frac{vol(\partial_r R_i)}{vol(R_i)} = 0$$

M is called *amenable* iff there is a regular sequence in M.

Theorem 1.4. *([BW2]) The following are equivalent:*

1. *M is non-amenable.*
2. *$H_0^{uf}(M) = 0$.*
3. *For all S uniformly discrete, $[S] = 0$.*

This means, in particular, that if, in the context of Theorem 1.2, M is non-amenable then for any N and S, $M \#_S N$ admits a metric of positive scalar curvature. If M is the universal cover of a compact manifold then amenability is equivalent to amenability of the fundamental group. Thus, for example, let X be a surface of higher genus cross S^4 connected sum any N. If $\hat{A}(N) \neq 0$ then X does not admit a metric of positive scalar curvature, but the universal cover does have a metric of uniformly positive scalar curvature in the bounded distortion class of the periodic metrics.

Naturally one wants to know whether $[S]$ in H_0^{uf} really is an obstruction to a metric of positive scalar curvature. That it is follows from the theory of the next section.

2 The uniformly finite $\hat{A}$ genus

How does one obstruct metrics of positive scalar curvature? In the compact case one has the $\hat{A}$ genus. We want to generalize this to the bounded geomtry setting.

According to Chern-Weil theory (see, for example, [LM]), the $\hat{A}$ class can be defined as the cohomology class of universal polynomial in the curvature tensor. On a manifold of bounded geometry, this form is bounded. If the metric is varied, Chern-Weil theory gives an explicit form whose differential is the change in $\hat{A}$ forms. If the change in metric is bounded distortion, this form is itself bounded. In this way one has a well defined element of l^∞-cohomology.

This fits nicely with the previous discussion as l^∞ cohomology is naturally Poincare dual to uniformly finite homology ([AW]). Thus we define the unformly finite $\hat{A}$ genus, $\hat{A}^{uf}$, as the dual of this form in H_0^{uf}. For the infinite connected sums we have been discussing, one has $\hat{A}^{uf}(M\#_S N) = \hat{A}^{uf}(M) + \hat{A}(N)[S]$, where by $\hat{A}(N)$ is interpreted as an integer.

Theorem 2.1. *If M has non-negative scalar curvature then $\hat{A}^{uf} = 0$.*

The proof of this theorem is, in outline, much like the corresponding theorem in the compact case: relate the $\hat{A}$ class to the index of a Dirac operator via the index theorem and then prove the vanishing of this index by a Bochner type argument. Both steps are substantially more difficult here, and some essentially new ingredients are needed.

We first need to reinterpret Lemma 1.3 cohomologically. Cheeger and Gromov ([CG1]) prove a chopping theorem for manifolds of bounded geometry which says that for any n there are constants $C_0, C_1, ..., C_n$ and r so that for any $S \subset M$ there is a codimension 0 manifold with boundary $(X, \partial X)$ such that:

1. $S \subset X \subset N_r(S)$
2. For $i = 0, 1, ..., n$ $\nabla^i II_{\partial X} \leq C_i$ where $II_{\partial X}$ is the second fundamental form, and ∇^i is the i^{th} covariant derivative.
3. $\frac{Vol(\partial X)}{Vol(\partial_r(S))}$ is bounded above and below independent of S.

Lemma 2.2. *For any n there are constants $C_0, C_1, ..., C_n$ and r so that if $\omega \in \Omega^m(M)$ then ω is d of a bounded form iff for some C*

$$|\int_X \omega| \leq Cvol(\partial X)$$

for all $(X, \partial X)$ compact, codimension 0 submanifold with $\nabla^i II_{\partial X} \leq C_i$ for $i = 0, 1, ..., n$.

Proof. The condition is necessary by Stokes theorem. To see that it is sufficient we use Lemma 1.3. Indeed, that lemma, formulated cohomologically, says that we need to show:

$$|\int_S \omega| \leq Cvol(\partial_r S)$$

With an arbitrary $S \subset M$ in place of X.

For any such S, approximate it by a manifold with boundary, X, via the chopping theorem above.

By (1) $|\int_S \omega|$ and $|\int_X \omega|$ differ by at most $||\omega|| Vol(\partial_r S)$ which, by part (3) of the chopping theorem, is bounded above by $Kvol(\partial X)$ for some K which depends on ω and M, but not on S.

Thus the bound on $|\int_S \omega|$ for arbitrary S follows, with perhaps a larger C, from the bound for submanifolds with bounded second fundamental forms.

□

In view of Lemma 2.2, Theorem 2.1 will follow from:

Theorem 2.3. *Let $(X, \partial X)$ be a compact spin manifold of non-negative scalar curvature. There is a C depending only on the curvature and second fundamental form so that*

$$\int_X \hat{A} \leq Cvol(\partial X)$$

We prove Theorem 2.3 by a detailed analysis of the Dirac operator on a manifold with boundary.

3 Index Theory

Let D be the canonical Dirac operator on spinors (much of this section works for an arbitrary geometric operators, but we will not need this generality).

Theorem 3.1. *([APS]) For any geometric operator D, there is a characteristic form ω, a polynomial P, and $n \in \mathbb{N}$, so that for any $(X, \partial X)$ compact we have:*

$$index(D) = \int_X w + \int_{\partial X} P(II_{\partial X}, \nabla II_{\partial X}, ..., \nabla^n II_{\partial X}) + \eta(\partial X)$$

Where $\eta(N) = \lim_{s \to 0^+} \Sigma_{\lambda \in Spec(D_\partial)} \lambda^{-s}$

We will use this to prove vanishing of ω in l^∞-cohomology via Lemma 2.2.

Given the bounds on $\nabla^i II_{\partial X}$, the middle term in Theorem 3.1 is bounded by a constant multiple of $Vol(\partial X)$.

Likewise, η is bounded linearly in $Vol(\partial X)$. This is shown for the signature operator in [CG2] and for a wide range of geometric operators including the Dirac operator on spinors in [Ra].

In view of this, and Lemma 2.2, we can interpret 3.1 as saying that the characteristic form in H_0^{uf} is a "uniformly finite index" of our operator.

For the Dirac operator on spinors, the form ω is the $\hat{A}$ form. The Dirac operator is related to positive scalar curvature by:

Theorem 3.2. *(Lichnerowicz formula, [LM])*

$$D^2 = \nabla^* \nabla + \frac{\kappa}{4}$$

Where ∇ is the canonical spinor connection, and κ the scalar curvature.

So, if s is a harmonic spinor (meaning $Ds = 0$) on X, we have:

$$0 = < \nabla^* \nabla s, s > + \frac{\kappa}{4} ||s||^2$$

If X were closed, we could integrate over X, and the first term on the RHS would be $||\nabla s||^2$.

Then, if $\kappa > 0$ both terms on the RHS would be ≥ 0, and therefore 0. This would mean $s = 0$. i.e. that there are no harmonic spinors, so that the index would be zero. Then by the Atiyah-Singer index theorem the $\hat{A}$ genus would be 0.

It is this argument we try to extend to our setting.

When one integrates the Lichnerowicz formula over an X with boundary there is an extra term which comes from the boundary term of integration by parts (see [LM]).

$$\int_X < \nabla^* \nabla s, s > = ||\nabla s||^2 - \int_{\partial X} < \nabla_\nu s, s >$$

(ν is the unit normal vector to ∂X):

The second term on the RHS is introduced when we integrate by parts.

So, when we have boundary, the Lichnerowicz formula becomes:

Lemma 3.3. *If s is a harmonic spinor on X, we have:*

$$0 = ||\nabla s||^2 + \int_X \frac{\kappa}{4} ||s||^2 - \int_{\partial X} < \nabla_\nu s, s >$$

If we assume the scalar curvature is ≥ 0 then the first two terms are non-negative. This can only happen if:

$$\int_{\partial X} < \nabla_\nu s, s > \geq 0$$

We can expand D in normal coordinates around the boundary:

Lemma 3.4. *([Gi]) Along ∂X we have:*

$$D = G(D_{\partial X} - \nabla_\nu - \frac{1}{2}tr(II))$$

Where G is the bundle automorphism induced by clifford multiplication by the normal vector, and $D_{\partial X}$ is the Dirac operator intrinsic to ∂_X.

$Ds = 0$ gives

$$< \nabla_\nu s, s > = < D_{\partial X} s, s > -\frac{1}{2}tr(II)||s||^2$$

Thus we have proven:

Proposition 3.5. *If X has non-negative scalar curvature, and s is a harmonic spinor, then:*

$$\int_{\partial X} < D_{\partial X} s, s > \geq \int_{\partial X} \frac{1}{2}tr(II)||s||^2$$

The boundary conditions for harmonic spinors in the Atiyah-Patodi-Singer index theorem are that when $s|_{\partial X}$ is expanded in eigenfunctions of $D_{\partial X}$, only negative eigenvalues are used.

Writing s as $\Sigma_\lambda a_\lambda s_\lambda$, where the s_λ are the eigenvectors of $D_{\partial X}$, 3.5 becomes:

$$\Sigma_\lambda \lambda ||a_\lambda||^2 \geq \frac{1}{2}\int_{\partial X} tr(II)||s||^2$$

Since all the λ must be negative, we must have some $\lambda \geq \lambda_0 = inf(\frac{1}{2}tr(II))$. By projecting onto these eigenspaces between λ_0 and 0, we get:

Theorem 3.6. *If $(X, \partial X)$ is spin and has non-negative scalar curvature then there is Λ, depending only on the second fundamental form, such that $dim(H) \leq N_{D^2_{\partial X}}(\Lambda)$, where H is the space of harmonic spinors on X with A-P-S boundary values, and $N_{D^2_{\partial X}}(\Lambda)$ is the dimension of the space of eigenfuctions of $D^2_{\partial X}$ below Λ.*

Theorem 3.7. *If N^n is a compact spin manifold, then for each λ there is a C_λ depending only on the curvature and injectivity radius of N, for which $N_{D^2}(\lambda) \leq C_\lambda vol(N)$*

Proof. By ([Bo],Prop 4.20(ii)) there are A and B so that $\lambda_n(D^2) \geq \lambda_{An-B}(\Delta)$, where Δ is the laplacian on functions. Thus the bound we need follows immediately from the same statement (with different C_λ) for the laplacian on functions.

Theorem 3.8. *([Gr]) There is a constant K depending only on curvature bounds and dimension for which for any compact manifold N we have the bound:*

$$\lambda_{V(\varepsilon)} \geq K\varepsilon^{-2}$$

For any $\varepsilon \leq inj\ rad$. Here $V(\varepsilon)$ is the minimal number of ε balls which cover N.

Since we have bounds on the curvature, the Heintz-Karcher comparison theorem ([Gr]) gives an L such that:

$$Vol(B(\varepsilon)) \geq L\varepsilon^n$$

for $B(\varepsilon)$ any such ε ball in N.

Choose a maximal family of disjoint $\frac{\varepsilon}{2}$ ball in N. By maximality the concentric balls of radius ε cover. But by disjointness there are at most

$$\frac{2^n Vol(N)\varepsilon^{-n}}{L}$$

balls. Therefore we have:

Proposition 3.9. *There is a constant C depending only on the curvature such that, for any $\varepsilon \leq inj\ rad(N)$*

$$\lambda_{Cvol(N)\varepsilon^{-n}} \geq \epsilon^{-2}$$

Turning this around to an upper bound on the spectral counting function gives:

Proposition 3.10. *There is a constant C depending only on the curvature, and a λ_0 depending only on curvature and injectivity radius, such that for any $\lambda \geq \lambda_0$*

$$N(\lambda) \leq CVol(N)\lambda^{\frac{n}{2}}$$

This gives the C_λ we needed for 3.7
$C_\lambda = C\lambda^{\frac{n}{2}}$ works for $\lambda \geq \lambda_0$.
and
$C_\lambda = C\lambda_0^{\frac{n}{2}}$
works for $\lambda \leq \lambda_0$, as
$N(\lambda) \leq N(\lambda_0)$

$\square$

This completes the proof of Theorem 2.3, and thereby Theorem 2.1.

4 Applications

Our first corollary, combined with the work of [BW2], gives a complete characterization of infinite connected sums of positive scalar curvature.

Corollary 4.1. *Let M, N, and S be as before. If $\hat{A}(N) \neq 0$ then $M\#_S N$ admits a metric of unformly positive scalar curvature iff $[S] = 0$ in $H_0^{uf}(M)$.*

In fact, 2.1 shows that if $[S] \neq 0$ then $M\#_S N$ does not even admit a metric of non-negative scalar curvature. Thus there is an interesting alternative: an infinite connected sum where M has uniformly positive scalar curvature either has a metric of uniformly positive scalar curvature or does not even have a metric of non-negative scalar curvature. It would be interesting to understand what happens when M only has non-negative scalar curvature. Theorem 2.1 still gives $[S]$ as an obstruction, but is very likely no longer sharp. As noted in the sketch of the proof of theorem 1.2, the construction there does not give metrics of bounded geometry unless one has a positive lower bound on κ. It seems likely that in the place of l^∞-cohomology one needs forms which go to zero at ∞ in some way related to κ. A closer examination of the behavior of C in Theorem 2.3 in terms of the lower bound on κ might give the right decay condition.

One intriguing aspect of Theorem 2.1 is that the obstruction lives in a non-Hausdorff group. This prevents the problem from fitting in to the C^* algebra framework usually used for these types of problems. To get around this, one can work with reduced uniformly finite homology, where the groups are the quotients of cycles by the closure of the boundaries. The reduced invariant is shown to obstruct positive scalar curvature in [Ro] and [BW2]. Corollary 4.1 shows that this loses important information. The necesity of working with non-Hausdorff groups makes it unclear how to get versions of the theorems for families, and in particular, it is unclear when $(M\#_S N) \times \mathbb{R}^i$ carries a metric of positive scalar curvature.

Theorem 2.3 has applications to compact manifolds as well. Recall the theorem of [KW] which says that for any compact manifold there is a metric whose non-positive scalar curvature is contained in an arbitrary ball. This is a purely topological statement. As a corollary of 2.3, one can see that this ball cannot be arbitrarily small.

Theorem 4.2. *Let M be a compact spin manifold with $\hat{A} \neq 0$. For any bounds on the curvature, there is an $r > 0$ so that there is no metric on M with those bounds on curvature and whose non-positive curvature set is contained in a ball of radius r.*

Proof. For r small enough the ball is embedded. Further, one has bounds on the second fundamental form of the sphere in terms of r and the curvature of M. Applying Theorem 2.3 to the complement of the ball gives a bound, which goes to 0 with r, on the integral of the $\hat{A}$ class over the complement of the ball. As the curvature is bounded, the integral over the ball is bounded by a multiple of its volume. These must add to a non-zero integer, which is a contradiction for r sufficiently small.

$\square$

Also, many of the standard applications (see, for example, [LM]) of the Bochner method can be carried over to manifolds with boundary as well: flat manifolds have their signature bounded by a multiple of the volume of the boundary, likewise for the total Betti number of manifolds with positive definite curvature tensor. For submanifolds of Euclidean space or the sphere, these results follow easily just from Alexander duality.

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