UNIT LENGTH COORDINATES

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Let M be $\mathbb{R}^2$ equipped with a metric in which the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are unit length vectors with an angle of θ:

$$||\frac{\partial}{\partial x}|| = 1$$

$$||\frac{\partial}{\partial y}|| = 1$$

and

$$< \frac{\partial}{\partial x}, \frac{\partial}{\partial y} >= \cos\theta$$

1. Preliminaries

This form of metric on $\mathbb{R}^2$ is much more general than the warped product example, so we will not be able to get as precise an understanding. One useful point of view is the alternative coordinates:

$$u = \frac{1}{\sqrt{2}}(x+y)$$

$$v = \frac{1}{\sqrt{2}}(x-y)$$

Notice that then:

$$||\frac{\partial}{\partial u}||^2 = 1 + \cos\theta$$

$$||\frac{\partial}{\partial v}||^2 = 1 - \cos\theta$$

$$< \frac{\partial}{\partial u}, \frac{\partial}{\partial v} >= 0$$

We can also use this to make the comparison with the warped product more concrete - both are given by orthogonal coordinates, with one constraint on the lengths. However, in the warped product case we also assume the metric is constant in one of the coordinate direction, whereas here θ can vary arbitrarily.

A good exercise in computation is to see if you can figure out what, if any, restrictions this poses on the manifold. For instance, can you find coordinates of this form for $\mathbb{H}^2$?

Another point worth noting is that this metric need not be complete: for instance, is $\theta \to 0$ near infinity like $\frac{1}{r}$ then the line $x + y = 0$ (and all parallel lines) has finite length. Similarly, if θ approaches π fast enough, the lines parallel to $y = x$ are finite length. On the other hand, if θ is bounded away from 0 and π, the metric is complete (the u and v coordinates are particularly useful in making this clear). Can you determine precisely what conditions make the manifold complete?

2. The Levi-Civita Connection

Since we have $< \frac{\partial}{\partial x}, \frac{\partial}{\partial x} >$ and $< \frac{\partial}{\partial y}, \frac{\partial}{\partial y} >$ are constant, we get:

$$< D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}, \frac{\partial}{\partial x} >= 0$$
$$< D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}, \frac{\partial}{\partial x} >= 0$$
$$< D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}, \frac{\partial}{\partial y} >= 0$$
$$< D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}, \frac{\partial}{\partial y} >= 0$$

The middle two equations give that $D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0$. Since $< \frac{\partial}{\partial x}, \frac{\partial}{\partial y} >= \cos\theta$, we get:

$$< D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}, \frac{\partial}{\partial y} > + < \frac{\partial}{\partial x}, D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} >= -\theta_x \sin\theta$$
$$< D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}, \frac{\partial}{\partial y} > + < \frac{\partial}{\partial x}, D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} >= -\theta_y \sin\theta$$

In both cases we already know that one term is zero, so we get:

$$< D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}, \frac{\partial}{\partial y} >= -\theta_x \sin\theta$$
$$< \frac{\partial}{\partial x}, D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} >= -\theta_y \sin\theta$$

So, for $D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}$ we know the inner product with $\frac{\partial}{\partial x}$ is zero, and with $\frac{\partial}{\partial y}$ is $-\theta_x \sin\theta$. This means we have:

$$D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = -\frac{\theta_x}{\sin\theta}(-\cos\theta \frac{\partial}{\partial x} + \frac{\partial}{\partial y})$$

and similarly,

$$D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\frac{\theta_y}{\sin\theta}(-\cos\theta \frac{\partial}{\partial y} + \frac{\partial}{\partial x})$$

Alternatively, in u and v as coordinates:

$$\begin{aligned}
D_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} &= \frac{1}{2} D_{\frac{\partial}{\partial x}+\frac{\partial}{\partial y}}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) = \frac{1}{2}(D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} + D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}) \\
&= -\frac{(1-\cos\theta)(\theta_x + \theta_y)}{\sin\theta}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) \\
&= -\frac{(1-\cos\theta)\theta_u}{\sin\theta} \frac{\partial}{\partial u}
\end{aligned}$$

similarly,

$$D_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v} = -\frac{(1-\cos\theta)\theta_u}{\sin\theta} \frac{\partial}{\partial u}$$

and

$$D_{\frac{\partial}{\partial u}} \frac{\partial}{\partial v} = \frac{(1+\cos\theta)\theta_v}{\sin\theta} \frac{\partial}{\partial v}$$

3. Geodesics

We can work in either coordinate system. First, in x and y. Let $c(t) = (x(t), y(t))$, and expand the equation $D_{c'} c' = 0$ as before to get:

$$x'' \frac{\partial}{\partial x} + y'' \frac{\partial}{\partial y} + 2x'y' D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} + x'^2 D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} + y'^2 D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = 0$$

substituting the calculations from the last section gives:

$$x'' - x'^2 \frac{\cos\theta}{\sin\theta}\theta_x - y'^2 \frac{1}{\sin\theta}\theta_y = 0$$

and

$$y'' - x'^2 \frac{1}{\sin\theta}\theta_x - y'^2 \frac{\cos\theta}{\sin\theta}\theta_y = 0$$

If u and v, if we write $c(t) = (u(t), v(t))$, we get :

$$u'' \frac{\partial}{\partial u} + v'' \frac{\partial}{\partial v} + 2u'v' D_{\frac{\partial}{\partial u}} \frac{\partial}{\partial v} + u'^2 D_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} + v'^2 D_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v} = 0$$

which gives the system:

$$u'' - (u'^2 + v'^2)\frac{1-\cos\theta}{\sin\theta}\theta_u = 0$$

and

$$v'' + 2u'v'\theta_v \frac{1+\cos\theta}{\sin\theta} = 0$$

It is easy to check that these are the same equations under the coordinate transforms. As we stated at the beginning, this metric is quite general, and there is not much to be said about the solutions to these equations without more information. Under further assumptions things can be easier - for instance, if θ depends only on u then the second equation says v is linear. If $v = At + B$ the first equation is then just $u'' - (u'^2 + A^2)\frac{1-\cos\theta}{\sin\theta}\theta_u = 0$, which you should be able to solve (see the discussion of metrics of constant curvature later for some places where this might be relevant).

4. Curvature

Since the manifold is 2-dimensional, there is only one two plane at each point, so curvature is described by a function κ giving the sectional curvature of these planes.

$$\kappa = \frac{< R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\frac{\partial}{\partial y}, \frac{\partial}{\partial x} >}{||\frac{\partial}{\partial x}||^2||\frac{\partial}{\partial y}||^2 - < \frac{\partial}{\partial x}, \frac{\partial}{\partial y} >^2}$$

The denominator is $1 - \cos^2\theta = \sin^2\theta$. For the numerator:

$$R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\frac{\partial}{\partial y} = D_{\frac{\partial}{\partial x}} D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} - D_{\frac{\partial}{\partial y}} D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} - D_{[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]} \frac{\partial}{\partial y}$$

The last term is zero because x and y are coordinates, and the middle term is zero because $D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = 0$, Using our calculation of $D_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}$, we get:

$$R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\frac{\partial}{\partial y} = D_{\frac{\partial}{\partial x}}(-\frac{\theta_y}{\sin\theta}(-\cos\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial x}))$$

by the Leibnitz rule, this is:

$$= (-\frac{\theta_x y}{\sin\theta} + \frac{\theta_y\theta_x\cos\theta}{\sin^2\theta})(-\cos\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial x})$$
$$-\frac{\theta_y}{\sin\theta} D_{\frac{\partial}{\partial x}}(-\cos\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial x})$$

Again using the Leibnitz rule and our calculation of the connection:

$$= (-\frac{\theta_{xy}}{\sin\theta} + \frac{\theta_y\theta_x\cos\theta}{\sin^2\theta})(-\cos\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial x})$$

$$-\frac{\theta_y}{\sin\theta}(\sin\theta\theta_x\frac{\partial}{\partial y} - \frac{\theta_x}{\sin\theta}(-\cos\theta\frac{\partial}{\partial x} + \frac{\partial}{\partial y})$$

$$= -\frac{\theta_{xy}}{\sin\theta}(\frac{\partial}{\partial x} - \cos\theta\frac{\partial}{\partial y})$$

so $< R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\frac{\partial}{\partial y}, \frac{\partial}{\partial x} >= -\theta_{xy}\sin\theta$, which gives:

$$\kappa = -\frac{\theta_{xy}}{\sin\theta}$$

We could instead use u and v coordinates and our calculation of the inner product there. We get the same answer, but expressed in u and v coordinates. This looks like:

$$\kappa = -\frac{\theta_{uu} - \theta_{vv}}{2\sin\theta}$$

Constant curvature

When does M have constant curvature? This means solving the equation:

$$\theta_{xy} = -\kappa\sin\theta$$

By reparametrizing, we may assume $\kappa = 0, 1$, or $, -1$.

We look at the cases separately:

Case 1: $\kappa = 0$

This reduces to the equation $\theta_{xy} = 0$. This says that $\frac{\partial}{\partial x}\theta$ is constant in y and vice-versa, so that θ is the sum of a function of x and a function of y:

$$\theta(x, y) = \theta_1(x) + \theta_2(y)$$

When a metric of this form is complete (can you figure out when this happens?) then we know M must be isometric to $\mathbb{R}^2$ - can you find the isometry? Can you describe the what the incomplete examples look like?

$\kappa \neq 0$

Here we get the equation $\theta_{xy} = \pm\sin\theta$. This equation becomes a bit easier if we switch to u and v as coordinates, so that we have:

$$\theta_{uu} - \theta_{vv} = \pm 2\sin\theta$$

The full solution here requires some more advanced knowledge of differential equations, but some solutions are within easy reach. If we look for a solution that is a function u only, we get a one variable equation:

$$\theta'' = \pm 2\sin\theta$$

or

$$\theta''\theta' = \pm 2\sin\theta\theta'$$

integrating,

$$\theta'^2 = \pm 4\cos\theta + A$$

for some constant A so that:

$$\theta' = \sqrt{A \pm 4\cos\theta}$$

If we let $F_A(t)$ be the integral of $\frac{1}{\sqrt{A\pm 4\cos\theta}}$, the solutions to this satisfy:

$$F_A(\theta) = u + B$$

for some constant B, so:

$$\theta = {F_A}^{-1}(u + B)$$

for some A and B. We cannot express F_A in closed form in general, but you should be able to calculate F_4 using double angle formula. (You'll get expressions like $\tanh^{-1}(\csc\frac{t}{2})$).

I'll stop here, there are some natural questions to address:

- Are these metrics of constant curvature complete? Are they defined everywhere? For warped products we saw that there are everywhere defined metrics of constant negative curvature, but not of constant positive curvature.
- More generally, can you find any complete metric of positive curvature defined on all of $\mathbb{R}^2$? We proved there is no warped product example. Here we're looking for a function θ which stays in the range $(0, \pi)$ and defined on all of $\mathbb{R}^2$ for which $\theta_{xy} < 0$. That's not quite enough, because we also would need to make sure the metric is complete. Can you sort this all out?