Worksheet 2 – Limits

16 January 2020

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(1) Answer the following questions with either "True" or "False". In your group, discuss how you would justify each answer.

(a) If $\lim_{x\to 0} f(x) = 1$, then $\lim_{x\to 0} (f(x) - 1) = 0$.

(b) If $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} (2f(x)) \geqslant \lim_{x\to 0} f(x)$.

(c) For any two functions f and g, $\lim_{x\to 5} f(x) + \lim_{x\to 5} g(x) = \lim_{x\to 5} (f(x) + g(x))$.

(d) If $\lim_{x\to -2^+} f(x) \neq \lim_{x\to -2^-} f(x)$, then $f(-2)$ is not defined.

(2) For the following piecewise function, state whether or not the limits exist, and if they do, find what they are. Be sure to show your work!

$$f(x) = \begin{cases} x^2 + 1 & x < 0 \\ 1 - x & 0 \leq x < 3 \\ -1 & x \leq 3 \end{cases}$$

(a) $\lim_{x\to -1} f(x)$

(b) $\lim_{x\to 0} f(x)$

(c) $\lim_{x\to 2} f(x)$

(d) $\lim_{x\to 3} f(x)$

(3) Use a table of values to determine whether or not $\lim_{x\to 0^+} \sin(\frac{1}{x})$ exists. (Hint: try using progressively smaller fractions $\frac{2}{\pi}, \frac{2}{2\pi}, \frac{2}{3\pi}, \frac{2}{4\pi}, \ldots$)

Once you've done this, try graphing $\sin(\frac{1}{x})$ (Desmos is a great free online graphing tool) to see visually why you get that answer.

(4) (a) Give the equation of two different functions for which the limit does not exist at one point (the point may be different for each function).

(b) Multiply the two functions you found in part (a) together.

(i) Does this function have points where the limit does not exist? If yes, are they the same points as in (a)?

(ii) Can you come up with two functions in part (a) such that their product does not have any points where the limit does not exist?

(c) Repeat part (b) with addition instead of multiplication. What happens if the points where the limits do not exist for each function must be different?

(Challenge) In 180, we define $\lim_{x \to a} f(x)$ to be a number L if we can make the values of $f(x)$ arbitrarily close to L by choosing x close enough to a (but not actually equal to a). Limits will be a crucial concept to develop all the ideas later in calculus. For the purposes of 180, the definition we just gave is good enough. However, it is often the case in higher-level math that we need to be a little more precise. Here is the "proper" definition of a limit:

Definition. The limit $L = \lim_{x \to a} f(x)$ exists if for any positive number ϵ, there is another positive number δ such that whenever x is a real number and $|x - a| < \delta$, then $|f(x) - L| < \epsilon$.

(5) Try to explain in plain English what this definition is saying and how it's related to the Math 180 definition (you don't need to write anything down, just think about it/discuss!)

(6) For each limit, and given ϵ, find a δ that works (don't try to do this until you can answer (5). Ask me if you need help with it!):

(a) $\lim_{x \to 0} x$ and $\epsilon = 1$

(b) $\lim_{x \to 0} 3x$ and $\epsilon = 1/2$

(c) $\lim_{x \to 0} x^2$ and $\epsilon = 1$

(d) $\lim_{x \to 0} x$ and $\epsilon = 1/2$

(e) $\lim_{x \to 5} 3x$ and $\epsilon = 1$

(f) $\lim_{x \to 1} \frac{1}{x}$ and $\epsilon = 1$

(g) **(Super Challenge)** In limits in (a),(b),(c), can you figure out what δ has to be in terms of ϵ to work for any choice of ϵ?