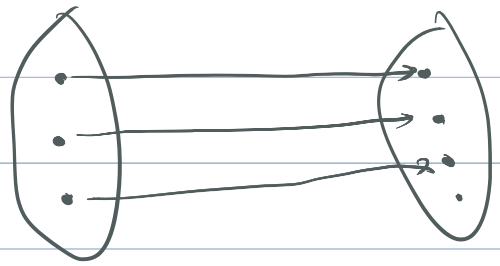
Math 294 Week 11 - Injections + Surjections

Defn A function $f : X \to Y$ is

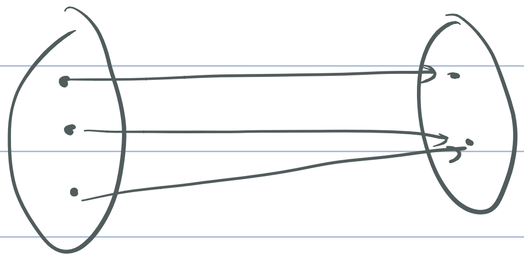
- **injective** if for all $x_1, x_2 \in X$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$
  (equivalently, $f(x_1) = f(x_2)$ implies $x_1 = x_2$)
- **surjective** if for all $y \in Y$, there is $x \in X$ such that $f(x) = y$
- **bijective** if it's both injective and surjective

eg $f : X \longrightarrow Y$



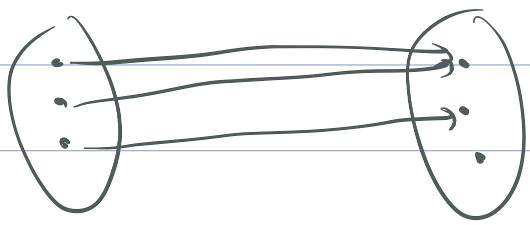
f is injective, but not surjective

$g : X \longrightarrow Y$



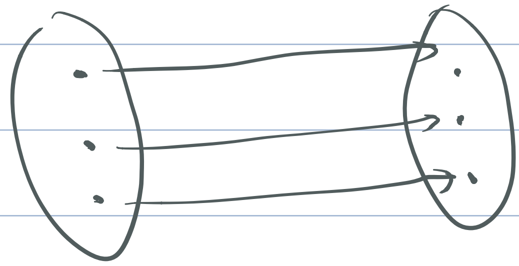
g is not injective but is surjective.

$h : X \longrightarrow Y$



h is neither injective nor surjective

$i : X \longrightarrow Y$



i is <u>both</u> injective and surjective (and thus is bijective)

Proving injections and surjections

Let $f: X \to Y$.

- To prove f is an injection, let $x_1, x_2 \in X$ and assume $x_1 \neq x_2$.
  Then prove that $f(x_1) \neq f(x_2)$.
  (Alternatively, assume $f(x_1) = f(x_2)$ and prove that $x_1 = x_2$)
- To prove f is <u>not</u> an injection, come up with two distinct elements $x_1, x_2 \in X$ for which $f(x_1) = f(x_2)$.

eg

- If $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 2x+1$, then f is injective.
  Let $x_1, x_2 \in \mathbb{R}$ and assume $f(x_1) = f(x_2)$.
  Then $2x_1 + 1 = 2x_2 + 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.
  Thus f is injective.
- If $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, then f is <u>not</u> injective, since $-1 \neq 1$ but $(-1)^2 = 1^2$.

- To prove f is surjective, let $y \in Y$ and find an appropriate $x \in X$ so that $f(x) = y$.
- To prove f is <u>not</u> surjective, find $y \in Y$ so that $f(x) \neq y$ for any $x \in X$.

eg • If $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 2x + 1$ then f is surjective.

Let $y \in \mathbb{R}$. Choose x to be $x = \frac{y-1}{2}$.

Then
$$f(x) = 2\left(\frac{y-1}{2}\right) + 1 = y - 1 + 1 = y.$$

• If $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, then f is <u>not</u> surjective, since there is no $x \in \mathbb{R}$ such that $x^2 = -1$.

Note that we showed $f(x) = 2x + 1$ is both injective and surjective, so it is bijective.

Function composition

Defn Let $f: X \to Y$ and $g: Y \to Z$.
The **composition** of f and g is a function $g \circ f: X \to Z$ defined by
$$(g \circ f)(x) = g(f(x))$$

eg $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 5$

$g : \mathbb{R} \to \mathbb{R}$ $\quad g(x) = x^2 + x + 1$

Then
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x+5) \\ &= (x+5)^2 + (x+5) + 1\end{aligned}$$

Theorem If $f : X \to Y$ and $g : Y \to Z$ are both injective, then $g \circ f$ is injective.

pf The domain of $g \circ f$ is X.

Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$.

We want to show $g(f(x_1)) \neq g(f(x_2))$.

Since f is injective and $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

Since g is injective and $f(x_1) \neq f(x_2)$,

$g(f(x_1)) \neq g(f(x_2))$, which is what we wanted. ∎

Theorem If $f : X \to Y$ and $g : Y \to Z$ are both injective, then $g \circ f$ is injective.

pf The codomain of $g \circ f$ is Z.

Let $z \in Z$. We need to find $x \in X$ such that $g(f(x)) = z$.

Since g is surjective, there is $y \in Y$ such that $g(y) = z$.

Since f is surjective, there is $x \in X$ such that $f(x) = y$.

Then $g(f(x)) = g(y) = z$, which is what we wanted. ⊠

Combining the previous two theorems gives us:

Theorem If $f: X \to Y$ and $g: Y \to Z$ are both bijective, then $g \circ f$ is bijective.