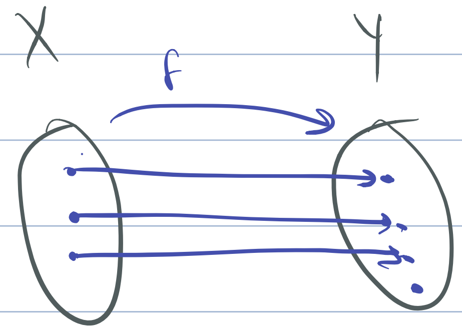
Math 294 Week 12 - Infinity

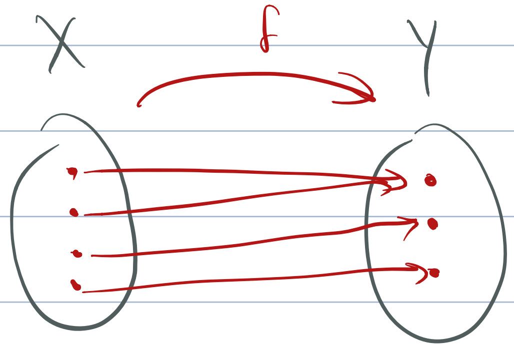
Recall

Def'n A function $f: X \to Y$ is

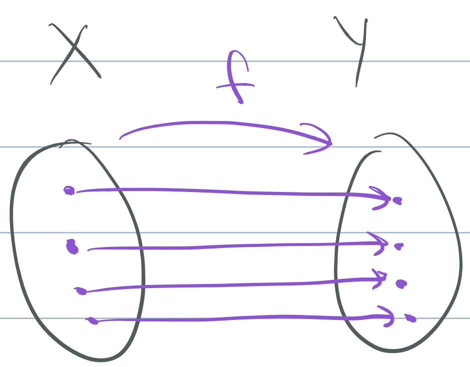
- **injective** if for all $x_1, x_2 \in X$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$



- **surjective** if for all $y \in Y$, there is $x \in X$ such that $f(x) = y$



- **bijective** if it is both injective and surjective.



Suppose X and Y are both finite,
and suppose we had a function $f: X \to Y$.

What can we say about the sizes of X and Y if:

- f is injective?
- f is surjective?
- f is bijective?

For any natural number $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, \dots, n\}$.

Defn A set X is **finite** if there is some $n \in \mathbb{N}$ and a bijection $f: [n] \to X$.
X is **infinite** if no such n exists.

Defn Given sets X and Y, we say

- X has **cardinality** less than or equal to Y, denoted $|X| \leq |Y|$, if there is an injection $f: X \to Y$.
- X has cardinality greater than or equal to Y ($|X| \geq |Y|$) if there is a surjection $g: X \to Y$.
- X has cardinality equal to Y ($|X| = |Y|$) if there is a bijection $h: X \to Y$.
- $|X| < |Y|$ means $|X| \leq |Y|$ but $|X| \neq |Y|$ and similarly for $|X| > |Y|$.

Def'n An infinite set X is countably infinite if there is a bijection $f : \mathbb{N} \to X$ (ie $|X| = |\mathbb{N}|$)

X is uncountably infinite if there is no such f.

Think of f as a way to list out the elements of X.

eg $\mathbb{Z}$ is countably infinite.

Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Theorem $|\mathbb{N}| < |\mathbb{R}|$ (the cardinality of $\mathbb{N}$ is strictly less than the cardinality of $\mathbb{R}$).

pf today's worksheet!