Math 294 Week 13 - Combinatorics

Q Given a group of n candidates, how many ways are there to choose a committee of k people?

Defn $\binom{n}{k}$ (read as "n choose k") denotes the number of ways to pick k elements from a set of n elements.

Q How many ways are there to choose a committee of 17 out of a group of 100 people with one chairperson?

Proposition $\binom{n}{k} = \binom{n}{n-k}$

pf Left hand side counts

Right hand side counts

Proposition $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

pf The left hand side can be written as

$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

This counts

The right hand side counts

Strategy (Proof by double counting)

To prove that two numbers are equal, it's enough to show that both numbers count the same finite set.

Strategy (Multiplication Principle)

Let X be a finite set.

Suppose that we have a step-by-step procedure for specifying the elements of X such that:

- each element is specified by a unique sequence of choices
- the choices available at each step depend only on previous steps
- the number of choices available at each step doesn't depend on the choices made

Then if there are n steps and m_k choices available at the k-th step,

$$|X| = \prod_{k=1}^{n} m_k = m_1 \cdot m_2 \cdot \dots \cdot m_k$$

Strategy (Addition principle)

A (finite) partition of a set X is a collection of subsets $U_1, \dots, U_n \subseteq X$ such that:

- each U_i is nonempty
- the U_i's are pairwise disjoint - that is,

  $U_i \cap U_j = \emptyset$ for $i \neq j$
- $U_1 \cup \dots \cup U_n = X$.

If $U_1, \dots, U_n$ is a partition of X, then $|X| = |U_1| + \dots + |U_n|$.