Quantifiers + Sets

Recall that last week, we introduced propositional logic as a way to formally express mathematical statements.
However, propositional logic isn't quite expressive enough for our needs - this is where quantifiers come in.

A brief interlude on sets

Sets are one of the most important ideas in mathematics.
Informally, a set is a collection of objects.
You've actually already seen countless examples of sets!

eg
- the set of natural numbers (denoted by $\mathbb{N}$)
- the set of rational numbers (denoted by $\mathbb{Q}$)
- the set of real numbers (denoted by $\mathbb{R}$)
- the set of fruits
- the set of US presidents
- the set of things in Kevin's fridge

We'll usually use capital letters to denote sets.

It's important to say if an object belongs to a particular set.

eg
- $\frac{1}{2}$ belongs to $\mathbb{Q}$
- $\frac{1}{2}$ is an element of $\mathbb{Q}$
- $\frac{1}{2} \in \mathbb{Q}$

- $\frac{1}{2}$ is not an element of $\mathbb{N}$

  $\frac{1}{2} \notin \mathbb{N}$

Quantifiers

In math, we often deal with variables:

eg "x is divisible by 2"

Quantifiers allow us to say something about what sort of values might satisfy a statement

Universal Quantification ("for all", $\forall$)

Let $P(x)$ be some statement involving the variable x, and let X be some set.

$\forall x \in X, P(x)$ means "for all x in X, $P(x)$ is true"

- eg "$\forall x \in \mathbb{N}$, x is divisible by 2" is false
- to prove a statement of the form "$\forall x \in X, P(x)$", assume that x is an element of X (but nothing else about x), then prove $P(x)$.
- eg Let E be the set of even integers. Show that $\forall x \in E$, $x^2 \in E$.

  proof Assume that $x \in E$ (so x is even).
  This means that $x = 2n$ for some integer n.
  Then $x^2 = (2n)^2 = 4n^2 = 2(2n^2)$
  Let $m = 2n^2$ (which is an integer).
  Since $x^2 = 2m$ where m is an integer, x^2 is even, and thus $x \in E$. ☒
- If in a proof you already know that "$\forall x \in X, P(x)$" is true, then if you know that $a \in X$, then you can conclude that $P(a)$ is true.

Existential Quantification ("there exists", $\exists$)

Let $P(x)$ be some statement involving the variable x, and let X be some set.

$\exists x \in X, P(x)$ means "there is x in X for which $P(x)$ is true"

- eg "$\exists x \in \mathbb{N}$, x is divisible by 2" is true
- to prove a statement of the form "$\exists x \in X, P(x)$", give an explicit example of an element $a \in X$ for which $P(a)$ is true.
- eg Show that $\exists n \in \mathbb{N}$, n is a perfect square and n is one more than a perfect cube.

  proof Let $n = 9$.
  Then $n = 3^2$ so n is a perfect square.
  Also, $n = 2^3 + 1$ so n is one more than a perfect cube.
- If in a proof you already know that "$\exists x \in X, P(x)$" is true, then you can select one element $a \in X$ for which $P(a)$ is true.

We can combine multiple quantifiers in a single statement - the order of the quantifiers can make a big difference in meaning!

- Let $p(x, y)$ mean "door x can be unlocked by key y".

  $\forall x \; \exists y \; p(x, y)$

  vs $\exists y \; \forall x \; p(x, y)$