Math 294 Week 4 - Sets

Naive definition of a set: a <u>set</u> is any collection of objects (possibly including other sets!)

Warning: this can cause problems!

New definition for a set:

Defn A **set** is a collection of objects (or <u>elements</u>) from a specified <u>universal set</u>,

We use $\mathcal{U}$ to denote the universal set.

Defn $x \in X$ means "x is an element of X".

$x \notin X$ means $\neg(x \in X)$, ie "x is <u>not</u> an element of X".

eg $\frac{1}{2} \in \mathbb{Q}$ (rational numbers)

$\frac{1}{2} \notin \mathbb{Z}$ (integers)

Specifying a set

Lists

- $\{1, 2, 3\}$
- $\{5, x, \text{apple}\}$
- $\{1, 2, \{1, 2\}\}$

Implied lists

- $\{1, 2, 3, \ldots, 20\}$
- $\{1, 2, 3, \ldots\}$
- $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$

Set-builder notation

Def'n Let X be a set, and $P(x)$ be some property.
Define $\{x \in X \mid P(x)\}$

as the set of elements x from X such that $P(x)$ is true.

eg
- $\{n \in \mathbb{N} \mid n \text{ is a prime number}\}$
- $\{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z},\ n = 2m\}$ = set of even integers
$= \{\ldots, -4, -2, 0, 2, 4, \ldots\}$
- $\{n^2 \mid n \in \mathbb{N}\} = \{m \mid \exists n \in \mathbb{N},\ m = n^2\}$

Proof strategy
To prove a statement of the form "$a \in \{x \in X \mid P(x)\}$", we need to prove that $a \in X$ and that $P(a)$ is true.

If we know that $a \in \{x \in X \mid P(x)\}$, we may assume $a \in X$ and $P(a)$ is true.

The empty set

Def'n The **empty set** is the set that has no elements.

We denote the empty set with $\emptyset$.

Note We should be careful about using the word "the", since it implies that there is only one empty set. This is true, but it's not part of the definition.

Subsets

Def'n Let X and Y be sets.

X is a **subset** of Y, denoted $X \subseteq Y$, if

$$\forall a, (a \in X \Rightarrow a \in Y) \quad [\forall a \in X, a \in Y]$$

$X \not\subseteq Y$ means X is not a subset of Y.

$X \subsetneq Y$ means X is a subset of Y, but is not equal to Y.

Proof strategy

To prove a statement of the form $X \subseteq Y$, take an arbitrary element $a \in X$ and show it is in Y.

If we know $X \subseteq Y$ and $a \in X$, then we can conclude $a \in Y$.

eg Let X be a set. Then $X \subseteq X$.

proof: We want to prove $\forall a \in X,\ a \in X$.

Let $a \in X$. Then $a \in X$ by assumption. ∎

eg $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

$\mathbb{N}$: natural numbers; $\mathbb{Z}$: integers; $\mathbb{Q}$: rational numbers; $\mathbb{R}$: real numbers; $\mathbb{C}$: complex numbers

Set equality

How do we say when two sets are equal?

A set is given by its elements, so we should say that two sets are equal when they contain exactly the same elements.

Axiom/Def'n (Set extensionality)

Let X and Y be sets.

Then $X = Y$ if and only if

$$\forall a,\ (a \in X \iff a \in Y)$$

Equivalently,

$$X \subseteq Y \quad \underline{\text{and}} \quad Y \subseteq X.$$

Proof strategy (Proof by double containment)

To prove $X = Y$,

- first prove $X \subseteq Y$

(let $a \in X$, prove $a \in Y$)

- then prove $X \supseteq Y$

  (let $a \in Y$, prove $a \in X$)

Set Operations

Intersection

Def'n The intersection of X and Y, denoted $X \cap Y$, is defined by

$$X \cap Y = \{ a \mid a \in X \wedge a \in Y \}$$

Proof strategy

To prove $a \in X \cap Y$, we prove that $a \in X$ and that $a \in Y$.

If we know $a \in X \cap Y$, then we know both $a \in X$ and $a \in Y$.

Union

Def'n The union of X and Y, denoted $X \cup Y$, is defined by

$$X \cup Y = \{ a \mid a \in X \vee a \in Y \}$$

Proof strategy

To prove $a \in X \cup Y$, we prove one of $a \in X$ or

$a \in Y$.

If we know $a \in X \cup Y$, we can split into two cases: one where $a \in X$ and one where $a \in Y$.

Complement

Def'n The relative complement of Y in X, denoted $X \setminus Y$, is defined by

$$X \setminus Y = \{ a \mid a \in X \land a \notin Y \}$$

Proof strategy

To prove $a \in X \setminus Y$, we prove $a \in X$ and $a \notin Y$.

If we know $a \in X \setminus Y$, we know both $a \in X$ and $a \notin Y$.

Def'n Let X be a set whose elements come from some universal set $\mathcal{U}$.

The complement of X, denoted X^c or $\overline{X}$, is defined by

$$X^c = \mathcal{U} \setminus X$$