Math 294 Week 5 - Sets II

Last week: introduced sets.

Recall

Subsets: $A \subseteq B$ means "$\forall a,\ a \in A \Rightarrow a \in B$"

Empty set: $\emptyset$ denotes the set with no elements.

The Power Set

Defn Let X be a set. The power set of X, denoted $\mathcal{P}(X)$, is the set of all subsets of X.

Symbolically, $\mathcal{P}(X) = \{ Y \mid Y \subseteq X \}$.

eg Let $X = \{1, 2\}$.

X has four subsets:

$\emptyset$, $\{1\}$, $\{2\}$, and $\{1,2\}$.

So $\mathcal{P}(X) = \{ \emptyset, \{1\}, \{2\}, \{1,2\} \}$.

eg Let X be any set.

Then $\emptyset \in \mathcal{P}(X)$ and $X \in \mathcal{P}(X)$.

pf First, we show $\emptyset \in \mathcal{P}(X)$.

To do this, we show $\emptyset \subseteq X$.

Let a be some element. We want to show $a \in \emptyset \Rightarrow a \in X$.

But $a \in \emptyset$ is false, so $a \in \emptyset \Rightarrow a \in X$ is true!

Thus $\emptyset \subseteq X$, and we conclude that $\emptyset \in \mathcal{P}(X)$.

Next, we show $X \subseteq X$.

We already did this last week, but for review:

Let $a \in X$. Then $a \in X$ by assumption, so $X \subseteq X$.

We conclude that $X \in \mathcal{P}(X)$. ∎

Key idea

For sets U and X,

$$U \subseteq X \iff U \in \mathcal{P}(X).$$

Warning! $\emptyset \subseteq \emptyset$ is <u>true</u>

$\Rightarrow \emptyset \in \mathcal{P}(\emptyset)$ is <u>true</u>

$\emptyset \in \emptyset$ is <u>false</u>

eg $X \subseteq Y \Rightarrow \mathcal{P}(X) \subseteq \mathcal{P}(Y)$.

pf Assume $X \subseteq Y$.

Let $U \in \mathcal{P}(X)$.

Then $U \subseteq X$.

Since $X \subseteq Y$, it follows that $U \subseteq Y$.

(exercise: write out a full proof of this step!)

So $U \in \mathcal{P}(Y)$.

Thus we conclude that $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$. ∎

Indexed families of sets

Defn Let I be some set (which we will call the *index set*).
An **indexed family of sets** is a specification of a set X_i for each $i \in I$.
We write $\{X_i \mid i \in I\}$ for the indexed family of sets.

A common index set is $\mathbb{N}$ (natural numbers):

eg For each $n \in \mathbb{N}$, let

$$X_n = \{0, \dots, n\}$$

So
$$X_0 = \{0\},$$
$$X_1 = \{0, 1\}$$
$$X_2 = \{0, 1, 2\}$$
$$\vdots$$

Defn The **indexed intersection** of an indexed family $\{X_i \mid i \in I\}$ is defined by

$$\bigcap_{i \in I} X_i = \{a \mid \forall i \in I,\ a \in X_i\}$$

The **indexed union** of an indexed family $\{X_i \mid i \in I\}$ is defined by

$$\bigcup_{i \in I} X_i = \{a \mid \exists i \in I,\ a \in X_i\}$$

eg Recall that if $a, b \in \mathbb{R}$, then $[a,b]$ is the interval between a and b:

$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

We can also replace "[" with "(" to change "$\leq$" to "$<$":

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$$
$$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$
$$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

Let $I = \{n \in \mathbb{N} \mid n \geq 1\} = \{1, 2, 3, \ldots\}$

For $n \in I$, let $X_n = [0, 1 + \frac{1}{n})$

Prove that $\displaystyle\bigcap_{n \geq 1} X_n = [0, 1]$

pf We proceed by double containment.

First, show $\displaystyle\bigcap_{n \geq 1} X_n \subseteq [0, 1]$

Let $a \in \bigcap_{n \geq 1} X_n$.

This means that $a \in [0, 1 + \frac{1}{n})$ for all $n \geq 1$.

In particular, $a \geq 0$.

Assume for the sake of contradiction that $a > 1$.

Then $a - 1 > 0$.

Let N be some natural number larger than $\frac{1}{a-1}$.

$\Rightarrow \frac{1}{N} \leq a - 1$

Thus $a \geq 1 + \frac{1}{N}$, and hence $a \notin [0, 1 + \frac{1}{N})$.

But we assumed that $a \in [0, 1 + \frac{1}{n})$ for all $n \geq 1$,

a contradiction!

So $a \le 1$ and therefore $a \in [0, 1]$.

Next, show $[0, 1] \subseteq \bigcap_{n \geq 1} X_n$

Let $a \in [0, 1]$.

That is, $0 \le a \le 1$.

To prove $a \in \bigcap_{n \geq 1} X_n$, we need to show that $a \in [0, 1 + \frac{1}{n})$ for all $n \geq 1$.

So fix $n \geq 1$.

Since $a \le 1 < 1 + \frac{1}{n}$, $a < 1 + \frac{1}{n}$.

Also, we know $a \geq 0$.

Thus $0 \le a < 1 + \frac{1}{n}$, and hence $a \in [0, 1 + \frac{1}{n})$.