Math 294 Week 6 – Contrapositive and Contradiction

Contrapositive

Def'n Consider a statement of the form $P \Rightarrow Q$.

- The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- The **inverse** of $P \Rightarrow Q$ is $\neg P \Rightarrow \neg Q$.
- The **contrapositive** of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$.

eg Fix two positive numbers m and n.

Consider the statement,

"If $mn > 64$, then either $m > 8$ or $n > 8$".

The converse is "If either $m > 8$ or $n > 8$, then $mn > 64$."

The inverse is "If $mn \leq 64$, then $m \leq 8$ and $n \leq 8$

The contrapositive is "If $m \leq 8$ and $n \leq 8$, then $mn \leq 64$.

Theorem The contrapositive of a statement is equivalent to the original statement.

Symbolically, $(P \Rightarrow Q)$ if and only if $(\neg P \Rightarrow \neg Q)$

pf We make a truth table:

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Proof strategy (Proof by Contrapositive)

To prove a statement of the form $P \Rightarrow Q$, it's enough to show $\neg Q \Rightarrow \neg P$, that is, assume Q is false and prove P is false.

eg Fix two positive numbers m and n.
Prove the statement,
"If $mn > 64$, then either $m > 8$ or $n > 8$".

pf We prove the contrapositive.
Assume that $m \leq 8$ and $n \leq 8$.
We want to show that $mn \leq 64$.
By algebra, $mn \leq 8 \cdot 8 = 64$, so we are done. ∎

eg Let $x \in \mathbb{Z}$. Prove that if $5x - 7$ is even, then x is odd.

pf We prove the contrapositive.
Assume that x is even.
Then $x = 2a$ for some integer a.
Thus
$$\begin{aligned} 5x - 7 &= 5(2a) - 7 \\ &= 10a - 7 \\ &= 10a - 8 + 1 \\ &= 2(5a - 4) + 1. \end{aligned}$$

Since $5a - 4$ is an integer, $5x - 7$ is odd. ∎

Contradiction

When proving things, we start with some assumptions and making logical deductions.
If we manage to prove something clearly false, assuming the reasoning is all correct, then something must have been wrong with our assumptions.

Proof strategy (Proof by contradiction)
To prove a statement P, you can first assume that P is false, then use this assumption to prove a contradiction.

Why does this work? Let C represent a contradiction (a false statement).
If we can prove $\neg P \Rightarrow C$, and C is false, then $\neg P$ must be false. So P is true.

eg There is no smallest positive real number.

pf Assume for the sake of contradiction that there is a smallest positive real number r.
Notice that $\frac{r}{2}$ is also a real number, and

$$0 < \frac{r}{2} < r,$$

This is a contradiction, since $\frac{r}{2}$ is a smaller positive real number.
So there is no smallest positive real number. ☺

eg Prove that $\sqrt{2}$ is irrational.

pf For this proof, we will need to know that n is even if and only if n^2 is even.

Assume for the sake of contradiction that $\sqrt{2}$ is rational. That is, there are integers a and b such that $\sqrt{2} = \frac{a}{b}$. In particular we can assume that $\frac{a}{b}$ is fully simplified, so a and b have no common factors.

Then

$$2 = \frac{a^2}{b^2}$$

$$\Rightarrow \quad 2b^2 = a^2$$

So a^2 is even.

This means that a is even.

So $a = 2c$ for some integer c.

Then

$$2b^2 = a^2 = (2c)^2 = 4c^2$$

$$\Rightarrow \quad b^2 = 2c^2$$

So b^2 is even, which means that b is even.

Hence both a and b are even.

But we assumed that $\frac{a}{b}$ was fully simplified — this is a contradiction!

So $\sqrt{2}$ is irrational. ∎

Note Some authors use ↯, $\rightarrow\leftarrow$, $\Rightarrow\Leftarrow$, or $\#$ to denote that a contradiction has been reached.

Thm (Euclid) There are infinitely many prime numbers.

pf Assume for the sake of contradiction that there are finitely many primes.

Call them $p_1, p_2, \ldots, p_n$ (a complete list).

Consider the number $q = p_1 p_2 \cdots p_n + 1$.

If q is composite, then it is divisible by some prime number.

However, notice that for each p_i, q divided by p_i has a remainder of 1.

So q cannot be composite, and thus it must be prime.

But q was not on our list of all primes, a contradiction!

So there must be infinitely many primes. ∎