# Math 294 Week 7 - Induction

**Theorem 1** For all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

But how do we prove such a thing?

**Theorem** (The Principle of Mathematical Induction)
For each $n \in \mathbb{N}$, let $P(n)$ be some statement.
If

1. $P(0)$ is true, and
2. For all $n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$ is true,

then for all $n$, $P(n)$ is true.

**Proof Strategy**
To prove a statement of the form $\forall n \in \mathbb{N}, P(n)$, it suffices to prove that $P(0)$ is true, and that for all $n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$ is true.

**Note** We can replace 0 with the smallest number we're trying to prove the statement for.

## Terminology

- The proof of $P(0)$ is called the base case.
- The proof of $\forall n, P(n) \Rightarrow P(n+1)$ is called the induction step.
- In the induction step, the assumption $P(n)$ is called the induction hypothesis.

eg

**Theorem 1** For all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

pf We proceed by induction on $n$.

Base case: we need to prove $P(1)$, that is,

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$$

LHS $= 1$ RHS $= \frac{2}{2} = 1$. ✓

Induction Step

Let $n \geq 1$, and assume that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ (induction hypothesis)

We want to prove that

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

We make the following calculations:

$$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \qquad \text{by induction hypothesis}$$

$$= \frac{n^2+n}{2} + \frac{2n+2}{2}$$

$$= \frac{n^2+3n+2}{2}$$

$$= \frac{(n+1)(n+2)}{2} \qquad \text{factoring}$$

So $P(n) \Rightarrow P(n+1)$.

Thus, by the principle of mathematical induction,

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

eg For all $n \geq 4$, we have $3n < 2^n$

pf We proceed by induction on $n$.

Base case We need to prove $P(4)$, that is,

$3 \cdot 4 < 2^4$. This is true since $3 \cdot 4 = 12 < 16 = 2^4$.

Induction Step

Let $n \geq 4$, and assume $3n < 2^n$. (IH)

We want to prove $3(n+1) < 2^{n+1}$.

$$3(n+1) = 3n+3$$

$$< 2^n + 3 \qquad \text{by IH}$$

$$< 2^n + 2^4$$

$$\leq 2^n + 2^n \qquad \text{since } n \geq 4$$

$$= 2 \cdot 2^n$$

$$= 2^{n+1}$$

So by the principle of mathematical induction,

for all $n \geq 4$, $3n < 2^n$. ■

non-eg Every horse is the same color.

The actual statement we "prove" is:

For all $n \geq 1$, if $X$ is a set of $n$ horses, then all horses in $X$ have the same color.

"proof"

Base Case Suppose there is just one horse.

The horse is the same color as itself, so the base case is true.

Induction Step

Let $n \geq 1$ and suppose that every set of $n$ horses is the same color. (IH)

Let $X$ be a set of $n+1$ horses.

If we remove the first horse from $X$, the last $n$ horses are the same color by IH.

If we remove the last horse from $X$, the first $n$ horses are the same color by IH.

So all horses in $X$ are the same color.

By induction, we're done. ■

What went wrong ?