Math 294 Week 8 - Induction II

Recall

To prove a statement of the form "$\forall n \in \mathbb{N}, P(n)$", it's enough to:

1. Prove $P(0)$
2. a) Let $n \in \mathbb{N}$, and assume $P(n)$
   b) Prove $P(n+1)$.

eg Recall that $\frac{d^n}{dx^n} f(x)$ means taking the derivative of $f(x)$ n times.

Also recall that $\frac{d}{dx} x^k = k x^{k-1}$

Prove that for all $n \geq 1$,

$$\frac{d^n}{dx^n} x^n = n! \quad (= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1)$$

pf We proceed by induction on n.

$P(n)$ is the statement "$\frac{d^n}{dx^n} x^n = n!$"

Base Case $P(1)$ is "$\frac{d}{dx} x = 1!$".

$\frac{d}{dx} x = 1$ and $1! = 1$, so $P(1)$ is true.

Induction Step Let $n \geq 1$, and assume that $P(n)$ is true.

That is, that $\frac{d^n}{dx^n} x^n = n!$. (IH)

We want to prove $P(n+1)$, that is,

$$\frac{d^{n+1}}{dx^{n+1}} x^{n+1} = (n+1)!$$

We calculate:

$$\frac{d^{n+1}}{dx^{n+1}} x^{n+1} = \frac{d^n}{dx^n} \left(\frac{d}{dx} x^{n+1} \right)$$

$$= \frac{d^n}{dx^n}\left((n+1)x^n\right)$$

$$= (n+1)\frac{d^n}{dx^n}x^n$$

$$= (n+1)\,n! \qquad \text{by IH}$$

$$= (n+1)!$$

So by induction,

$$\frac{d^n}{dx^n}x^n = n!$$

for all $n \geq 1$.

Strong Induction

Recall "regular" or "weak" induction:

Theorem (The Principle of Mathematical Induction)

For each $n \in \mathbb{N}$, let $P(n)$ be some statement.

If

1. $P(0)$ is true, and
2. For all $n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$ is true,

then for all n, $P(n)$ is true.

Sometimes, weak induction is not powerful enough.

We can make some changes (highlighted in red):

Theorem (The Principle of Strong Induction)

For each $n \in \mathbb{N}$, let $P(n)$ be some statement.

If

① $P(0)$ is true, and

② For all $n \in \mathbb{N}$, if $P(k)$ is true for all $k \leq n$, then $P(n+1)$ is true.

then for all n, $P(n)$ is true.

eg (First, we make the observation that $r = \frac{3+\sqrt{5}}{2}$ is an irrational number, and that $r + \frac{1}{r} = 3$).

Prove that if r is a nonzero real number such that $r + \frac{1}{r}$ is an integer, then for all integers $n \geq 1$, $r^n + \frac{1}{r^n}$ is an integer.

pf We proceed by strong induction on n.

$P(n)$ is the statement "$r^n + \frac{1}{r^n}$ is an integer".

<u>Base case</u> $P(1)$ is "$r^1 + \frac{1}{r^1}$ is an integer".

$r^1 + \frac{1}{r^1} = r + \frac{1}{r}$ which we assumed to be an integer.

So $P(1)$ is true.

<u>Induction Step</u>

Let $n \geq 1$.

Assume that $P(k)$ is true for all $1 \leq k \leq n$.

That is, that

$r^k + \frac{1}{r^k}$ is an integer for all $1 \leq k \leq n$. (IH)

We want to show that $r^{n+1} + \frac{1}{r^{n+1}}$ is an integer.

Notice that

$$r^{n+1} + \frac{1}{r^{n+1}} = \left(r^n + \frac{1}{r^n}\right)\left(r + \frac{1}{r}\right) - \left(r^{n-1} + \frac{1}{r^{n-1}}\right)$$

By the induction hypothesis, each of $\left(r^n + \frac{1}{r^n}\right)$, $\left(r + \frac{1}{r}\right)$, and $\left(r^{n-1} + \frac{1}{r^{n-1}}\right)$ are integers.

So $r^{n+1} + \frac{1}{r^{n+1}}$ is an integer as well, since multiplying and subtracting integers gives another integer.

Thus $P(n+1)$ is true.

By strong induction, if $r + \frac{1}{r}$ is an integer, then for any $n \geq 1$, $r^n + \frac{1}{r^n}$ is also an integer. ∎

__Note__ If we used weak induction, we wouldn't be able to say that $r^{n-1} + \frac{1}{r^{n-1}}$ is an integer in the induction step.

So we really do need to use strong induction here.