Math 294 Week 9 - Functions

Def'n Let X and Y be two sets.
A **function** f from X to Y is a specification of elements $f(x) \in Y$ for $x \in X$, such that

$$\forall x \in X, \ \exists! \, y \in Y, \ y = f(x)$$

exists unique (exactly one)

For $x \in X$, the (unique) element $f(x) \in Y$ is called the **value** of f at x.
The set X is called the **domain** of f.
The set Y is called the **codomain** of f.
We write $f : X \to Y$ to denote that f is a function from X to Y.

Functions are often described using a formula,
eg $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$,
but this doesn't have to be the case.
eg $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Functions often involve numbers, but this doesn't have to be the case.
eg $h : \{a, b, c\} \to \{\text{red}, \text{green}, \text{blue}\}$

defined by

$h(a) = $ red

$h(b) = $ blue

$h(c) = $ red

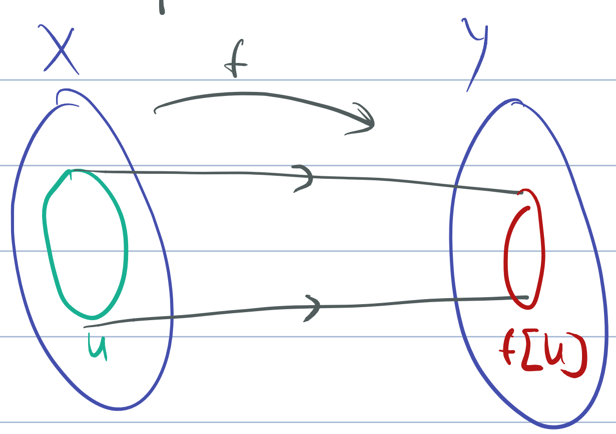
Images + Preimages

Def'n Let $f: X \to Y$ be a function and let $U \subseteq X$.

The image of U under f is the subset $f[U] \subseteq Y$ defined by

$$f[U] = \{ f(x) \mid x \in U \}$$
$$= \{ y \in Y \mid \exists x \in U,\ y = f(x) \}$$ (alternate definition)

In words, $f[U]$ is the set of values that f takes on inputs from U.



The image of f is the set $f[X]$, that is, the set of all possible values that f can take.

eg Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.

The image of f is the set of nonnegative real numbers $\mathbb{R}^{\geq 0}$.

pf We want to show $f[\mathbb{R}] = \mathbb{R}^{\geq 0}$.

We proceed by double containment.

($f[\mathbb{R}] \subseteq \mathbb{R}^{\geq 0}$)

Let $y \in f[\mathbb{R}]$.

This means that $y = f(x) = x^2$ for some $x \in \mathbb{R}$.

We know $x^2 \geq 0$ for any $x \in \mathbb{R}$, so $y \in \mathbb{R}^{\geq 0}$.

($\mathbb{R}^{\geq 0} \subseteq f[\mathbb{R}]$)

Let $y \in \mathbb{R}^{\geq 0}$.

Since y is nonnegative, it has a square root $\sqrt{y} \in \mathbb{R}$.

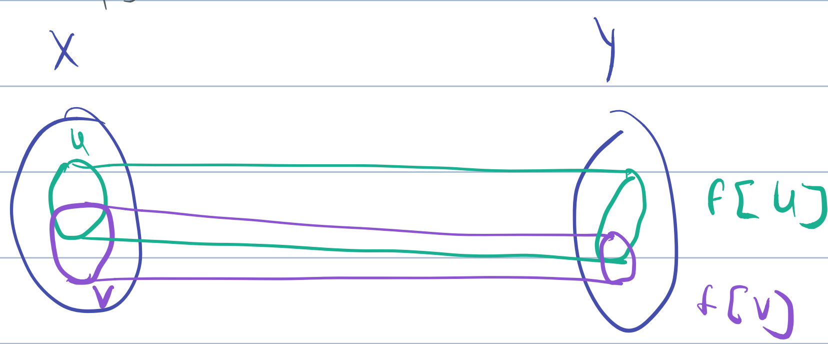
Then $y = (\sqrt{y})^2 = f(\sqrt{y})$.

So $y \in f[\mathbb{R}]$.

Thus by double containment, $f[\mathbb{R}] = \mathbb{R}^{\geq 0}$. ⊠

eg Let $f : X \to Y$ be a function, and let $U, V \subseteq X$.

Is it always true that $f[U \cap V] \subseteq f[U] \cap f[V]$?



pf Let $y \in f[U \cap V]$.

Then $y = f(x)$ for some $x \in U \cap V$.

By definition of intersection, $x \in U$ and $x \in V$.

We need to show $y \in f[U]$ and $y \in f[V]$.

Since $x \in U$ and $f(x) = y$, $y \in f[U]$.

Since $x \in V$ and $f(x) = y$, $y \in f[V]$.

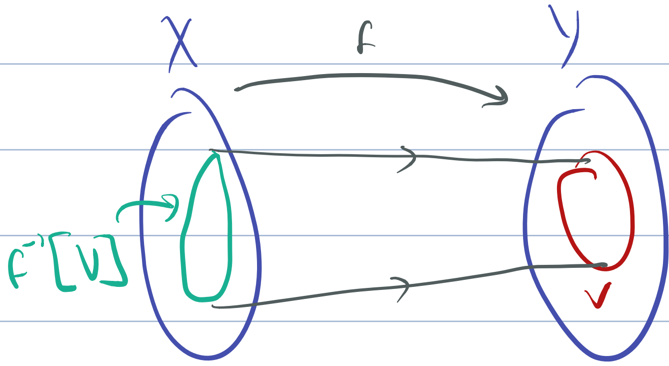
Hence by definition of intersection, $y \in f[U] \cap f[V]$.

Therefore, $f[U \cap V] \subseteq f[U] \cap f[V]$.

Defn Let $f : X \to Y$ be a function and let $V \subseteq Y$.
The *preimage of V under f* is the subset $f^{-1}[V] \subseteq Y$ defined by

$$f^{-1}[V] = \{ x \in X \mid f(x) \in V \}.$$

In words, $f^{-1}[V]$ is the set of elements in X that f sends to an element of V.



eg Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.

Then

- $f^{-1}[\{1, 4\}] = \{-2, -1, 1, 2\}$
- $f^{-1}[\{-1, 1, 4\}] = \{-2, -1, 1, 2\}$
- $f^{-1}[\{1, 2, 3, 4\}] = \{-2, -\sqrt{3}, -\sqrt{2}, -1, 1, \sqrt{2}, \sqrt{3}, 2\}$

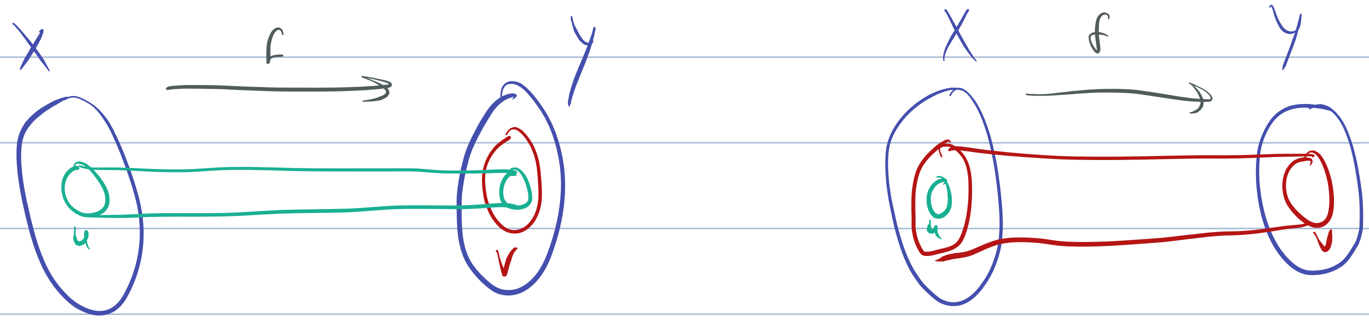
Let $g : \mathbb{Z} \to \mathbb{Z}$ be given by $g(x) = x^2$

Then

- $f^{-1}[\{1, 4\}] = \{-2, -1, 1, 2\}$
- $f^{-1}[\{1, 2, 3, 4\}] = \{-2, -1, 1, 2\}$

eg Let $f : X \to Y$ be a function, and let $U \subseteq X$ and $V \subseteq Y$.

Then $f[U] \subseteq V$ if and only if $U \subseteq f^{-1}[V]$.



pf ($f[U] \subseteq V \Rightarrow U \subseteq f^{-1}[V]$)

Assume that $f[U] \subseteq V$.

Let $x \in U$.

Then $f(x) \in f[U]$ by the definition of image.

Since $f[U] \subseteq V$, $f(x) \in V$.

So $x \in f^{-1}[V]$ by the definition of preimage.

As x was arbitrary, $U \subseteq f^{-1}[V]$.

($U \subseteq f^{-1}[V] \Rightarrow f[U] \subseteq V$)

Assume $U \subseteq f^{-1}[V]$.

Let $y \in f[U]$.

By definition of image, there is $x \in U$ such that $y = f(x)$.

Since $U \subseteq f^{-1}[V]$, $x \in f^{-1}[V]$.

By definition of preimage, $f(x) \in V$.

Since $y = f(x)$, $y \in V$.

As y was arbitrary, $f[U] \subseteq V$.