ESP Math 294 | **Week 1 - Solutions** | Spring 2021

Introduction & What is a proof?

January 11, 2021

Note: the proofs in these solutions are fairly detailed, just to give you an idea of all the things we can explain in your proofs. As the semester progresses and we become more familiar with the material, it'll be okay to provide slightly less detail.

For problems 1 and 2, we will need precise definition of even and odd numbers. We say that an integer n is an *even number* if there is another integer k such that $n = 2k$. We say that an integer n is an *odd number* if there is another integer k such that $m = 2k + 1$.

In order to prove that an integer n is even, we have to be able to write it in the form $n = 2k$, where k is also an integer. To prove that n is odd, we have to be able to write it in the form $n = 2k + 1$, where k is an integer.

1. Prove that the sum of two even numbers is an even number, and that the sum of two odd numbers is an even number.

   Proof. First, we prove that the sum of two even numbers is an even number. Let n and m be two even numbers. By the definition of an even number, there is an integer k such that $n = 2k$ and there is an integer l such that $m = 2l$. Then $n + m = 2k + 2l = 2(k + l)$. Since $k + l$ is an integer and $n + m = 2(k + l)$, the definition of an even number tells us that $n + m$ is even.

   Next, we prove that the sum of two odd numbers is an even number. Let n and m be two odd numbers (note that since this is a different part than the previous section, these are not the same n and m as before). By the definition of an odd number, there is an integer k such that $n = 2k+1$ and there is an integer l such that $m = 2l+1$. Then $n+m = 2k+1+2l+1 = 2k+2l+2 = 2(k+l+1)$. Since $k + l + 1$ is an integer and $n + m = 2(k + l + 1)$, the definition of an even number tells us that $n + m$ is even. □

2. Prove that if n is even, then n^2 is even, and if n is odd, then n^2 is odd.

   Proof. First, we prove that if n is even, then n^2 is even. Let n be an even number. By definition of an even number, there is an integer k such that $n = 2k$. Then $n^2 = 2k \cdot 2k = 2 \cdot 2k^2$. Since $2k^2$ is an integer and $n^2 = 2(2k^2)$, the definition of an even number tells us that n^2 is even.

   Next, we prove that if n is odd, then n^2 is odd. Let n be an odd number. Then there is an integer k such that $n = 2k + 1$. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k$ is an integer and $n^2 = 2(2k^2 + 2k) + 1$, the definition of an odd number tells us that n^2 is odd. □

3. If a and b are integers, we say that a *divides* b if there is an integer r so that $b = ar$ (think of this as saying that $\frac{b}{a}$ gives you a whole number). Prove that if a divides b and b divides c, then a divides c.

   Proof. Assume that a divides b and b divides c. By definition, this means that there is an integer r such that $b = ar$ and there is an integer s such that $c = bs$. Then substituting $b = ar$ into the second equation, we have that $c = (ar)s = a(rs)$ (remember that we are allowed to assign parentheses to multiplication however we want - this is called the *associative property*).

   Since rs is an integer, this shows that a divides c. □

4. Suppose that $a < b$. Prove that $a < \frac{a+b}{2} < b$. (Hint: you should prove that $a < \frac{a+b}{2}$ and that $\frac{a+b}{2} < b$ are true separately.)

   Proof. First, we prove that $a < \frac{a+b}{2}$. Notice that we can write a as $\frac{a}{2} + \frac{a}{2}$. Then since $a < b$, $\frac{a}{2} < \frac{b}{2}$. Hence

$$a = \frac{a}{2} + \frac{a}{2} < \frac{a}{2} + \frac{b}{2} = \frac{a+b}{2}.$$

   The proof that $\frac{a+b}{2}$ is similar: We can write b as $\frac{b}{2} + \frac{b}{2}$. Then

$$\frac{a+b}{2} = \frac{a}{2} + \frac{b}{2} < \frac{b}{2} + \frac{b}{2} = b.$$

   □

5. **(Challenge)** Prove that $\sqrt{2}$ is an irrational number (a *rational number* is a number that can be written as a fraction $\frac{p}{q}$, where p and q are both integers, while an *irrational number* is a number that, well, isn't rational).

   We'll see this proof later on in the course!