ESP Math 294 | Spring 2021

Week 11 Solutions

Injections, Surjections, & Bijections

March 28, 2021

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1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = 2n + 1$. Is f injective? Is f surjective? Prove your answers.

   Solution: f is injective: Let $n_1, n_2 \in \mathbb{Z}$. Suppose that $f(n_1) = f(n_2)$. We need to show that $n_1 = n_2$. Since $f(n_1) = f(n_2)$, we have $2n_1 + 1 = 2n_2 + 1$. Subtracting 1 on both sides gives us $2n_1 = 2n_2$, and dividing by 2 on both sides gives us that $n_1 = n_2$, as desired. So f is injective.

   f is not surjective: Consider $2 \in \mathbb{Z}$. For any $n \in \mathbb{Z}$, $2n + 1$ is going to be an odd integer, so there is no n such that $f(n) = 2$. This shows that f is not surjective.

   Given a set A, we can define a function $\mathrm{id}_A : A \to A$ by $\mathrm{id}_A(a) = a$ for every $a \in A$. The function id_A is called the *identity function*, and is the function that does nothing to its inputs.

2. Let A, B be nonempty sets and $f : A \to B$ be an injective function. Prove that there is a surjection $g : B \to A$ such that $g \circ f = \mathrm{id}_A$. ($g$ is called a *left inverse.*)

   Solution:

   We need to define g, that is, for every $b \in B$, we need to say that value $g(b)$ is. We define g as follows: fix some arbitrary $a_0 \in A$. For each $b \in B$,

   $$g(b) = \begin{cases} a & \text{if there is } a \in A \text{ such that } f(a) = b \\ a_0 & \text{otherwise} \end{cases}$$

   g is well-defined–since f is injective, there can be at most one value of a for which $f(a) = b$, and we have given a value of $g(b)$ for every single $b \in B$.

   Now, we show that g is surjective: let $a \in A$. We want to find $b \in B$ such that $g(b) = a$. Take the value of b to be $f(a)$. Since $f(a) = b$, our definition for g ensures that $g(b) = a$. So g is surjective. $\square$

   Finally, we show that $g \circ f = \mathrm{id}_A$. Let $a \in A$. We need to show that $g(f(a)) = a$. If we set $b = f(a)$, then by our definition for g, $g(b) = a$. So then $g(f(a)) = a$, exactly as desired. $\square$

3. Let A, B be nonempty sets and $f : A \to B$ be a surjective function. Prove that there is an injection $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. ($g$ is called a *right inverse.*)

4. Let A be any set, and $f : A \to A$. Prove that if f is injective but not surjective, then A must have infinitely many elements.