Week 12
Infinity

April 6, 2021

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1. In this problem, we will prove that $|\mathbb{N}| < |\mathbb{R}|$. This means that, in a very precise sense, there are "more" real numbers than natural numbers.
   (a) What two things do we need to show in order to prove that $|\mathbb{N}| < |\mathbb{R}|$?
   (b) Come up with an injection $g : \mathbb{N} \to \mathbb{R}$. Which of the two things from (a) does this show?
   (c) We'll now show that there cannot be a surjection from $\mathbb{N}$ to $\mathbb{R}$.
   Let $f : \mathbb{N} \to \mathbb{R}$ be any function. The goal is to show that f is not a surjection.
      i. What do we need to do in order to show that a function is not a surjection?
      ii. For each $n \in \mathbb{N}$, $f(n)$ is a real number. Every real number has a unique decimal expansion, so let $f(n)$ have decimal expansion $a_{n0}.a_{n1}a_{n2}a_{n3}\cdots$, where $a_{n0} \in \mathbb{Z}$ and each a_{ni} for $i \geq 1$ is a digit between 0 and 9, and the decimal expansion doesn't end with repeated 9's (for example, we write $1/2$ as $0.500\cdots$ instead of $0.499\cdots$).
      So we have
      $$\begin{aligned} f(0) &= a_{00}.a_{01}a_{02}a_{03}\cdots \\ f(1) &= a_{10}.a_{11}a_{12}a_{13}\cdots \\ f(2) &= a_{20}.a_{21}a_{22}a_{23}\cdots \\ &\vdots \end{aligned}$$
      and so on.
      Can you come up with a real number that doesn't appear as any of the $f(n)$'s? Think about this for a few minutes, and if you can't come up with anything, move on to the next parts of the problem.
      iii. We'll define a real number b as follows:
      b will have decimal expansion $b_0.b_1b_2b_3\cdots$, where
      $$b_i = \begin{cases} 0 & \text{if } a_{ii} \neq 0 \\ 1 & \text{if } a_{ii} = 0 \end{cases}.$$
      (For example, suppose that $f(0) = 0.12333\cdots$, $f(1) = 12.313131\cdots$, $f(2) = 4.4000\cdots$, and $f(3) = 0.450100\cdots$. Then we would set $b = 1.011\cdots$.)
      Is there an $n \in \mathbb{N}$ for which $f(n) = b$? If so, which n? If not, why not?
      iv. What does part iii. tell us about f? And what does that tell us about the relationship between $|\mathbb{N}|$ and $|\mathbb{R}|$?
2. Use a similar argument to prove that for any set X, $|X| < \mathcal{P}(X)$.
3. Prove that $\mathbb{Q}$ is countable. (Hint: first prove that $\mathbb{Z} \times \mathbb{Z}$ is countable, then construct a surjection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Q}$.)
4. David Hilbert is running a hotel, but his hotel is very special–it has infinitely many rooms, numbered 1, 2, 3, and so on.