ESP Math 294 | Spring 2021

Week 2 Solutions

Propositional Logic & Logic Puzzles

January 17, 2021

All groups:

1. Describe the steps that should be followed in order to prove statements of the following forms (don't use a truth table):

   (a) $(p \wedge q) \Rightarrow (\neg r)$

   (b) $(p \Rightarrow q) \iff (\neg q \Rightarrow \neg p)$

   (a) We first assume $p \wedge q$, which means that we can assume p is true and q is true. Then, we must prove $\neg r$, which means that we need to prove that r is false.

   (b) The statement $(p \Rightarrow q) \iff (\neg q \Rightarrow \neg p)$ is the same as

   $$((p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)) \wedge ((\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)).$$

   This means that we need to prove $(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)$ and $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$.

   To prove $(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)$, we assume that $p \Rightarrow q$ is true, and want to prove $\neg q \Rightarrow \neg p$. To prove this, we assume $\neg q$ (that is, q is false) and prove $\neg p$ (that is, p is false).

   The steps for $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$ are similar.

2. Fontano's is an Italian sandwich shop near campus. Let p mean "you get a bag of chips and a drink for free from Fontano's", q mean "you buy a sandwich from Fontano's", r mean "you buy a slice of pizza from Fontano's", and t mean "it's a Tuesday".

   What does the following propositional statement mean in plain English?

   $$t \wedge (q \vee r) \Rightarrow p$$

   (note that this is not actually true in real life)

   If you buy a sandwich or a slice of pizza from Fontano's on a Tuesday, you get a bag of chips and a drink for free.

3. Consider the statement "The Blue Line will be delayed if it has snowed more than one inch, and if the Blue Line is delayed, Kevin will be late for his class" (we're either in pre-COVID or post-COVID times for this problem, so Kevin actually has to go to campus). Write this statement as a propositional statement. Be sure to specify what each propositional variable represents.

   Let p mean "the Blue Line will be delayed", q mean "it has snowed more than one inch", and r mean "Kevin will be late for his class".

   The statement will be

   $$(q \Rightarrow p) \wedge (p \Rightarrow r).$$

4. For each of the following, find a propositional statement that matches the unlabeled column in the truth table:

   (a)

p	q	
T	T	F
T	F	T
F	T	T
F	F	F

There are many possible answers, but $(p \wedge \neg q) \vee (\neg p \wedge q)$ is probably the simplest.

(b)

p	q	
T	T	T
T	F	F
F	T	T
F	F	F

There are many possible answers, but the simplest is q.

Group 1:

5. Zoey says, "at least one of the following is true: I am a truth-teller, or Mel is a truth-teller." Mel says, "Zoey could say that I am a ~~truth-teller~~ liar." What is the identity of each person? Prove your answer first by a case analysis, and then by using a truth table.

Case analysis:

Before beginning the case analysis, it's probably helpful to decipher what Mel is saying. If Zoey *could* say that Mel is a liar, this actually means that Zoey *would* say that Mel is a liar – Zoey can only say one thing about Mel, and so if it is possible for her to say that thing, then it will be the only thing she says.

Case 1: Zoey is a truth-teller.

Then Zoey's statement is true, since the "I am a truth-teller" part is true. If Mel is a liar, this means that Zoey could not say that Mel is a liar, that is, Zoey will say that Mel is a truth-teller. Since Zoey is a truth-teller and Mel is a liar, this cannot happen.

If Mel is a truth-teller, then Zoey could say that Mel is a liar. Our reasoning above tells us that Zoey *would* say that Mel is a liar. Since Zoey is a truth-teller, then Mel is a liar, but this contradicts our assumption that Mel is a truth-teller. So this cannot happen.

Case 2: Zoey is a liar.

Then Zoey's statement is false, which means that both Zoey and Mel are liars.

So we come to the conclusion that both are liars, but it's a good idea to check that Mel being a liar works out. If Mel is a liar, then Zoey could not say that Mel is a liar, that is, Zoey would say that Mel is a truth-teller. This works out, since Zoey is a liar and Mel is a liar.

So the final conclusion is that both Zoey and Mel are liars.

Truth table:

Let p be "Zoey is a truth-teller" and q be "Mel is a truth-teller". Zoey's statement translates to $p \vee q$.

Mel's statement is a little harder, but it translates to $(p \Rightarrow \neg q) \wedge (\neg p \Rightarrow q)$ – at the beginning of the case analysis, we said that Mel's statement is that Zoey will say that Mel is a liar. So if Zoey is a truth teller (p), then Mel is a liar ($\neg q$). If Zoey is a liar ($\neg p$), then Mel is a truth-teller (q).

If you know about inverses, then you might recognize that Mel's statement is the same as $p \iff \neg q$ – it's fine if you don't, but we'll use this for simplicity in the truth table.

The truth table we get is the following:

p	q	Zoey says $p \vee q$	Mel says $p \iff \neg q$
T	T	T / T	F / T
T	F	T / T	T / F
F	T	T / F	T / T
F	F	F / F	F / F

where in the 3rd and 4th columns, the first value comes from the values of p and q, while the second value comes from the identity of the person making the statement.

Only the last row is consistent, so it must be that Zoey and Mel are both liars.

6. You are given two chests. You know that inside each chest is either a treasure or a deadly trap. On chest A, there is an inscription: "At least one of these chests contains a treasure." On chest B, there is an inscription: "Chest A contains a deadly trap." You also know that either both inscriptions are true, or both are false.

   Can you choose one of the chests to open to guarantee that you will receive a treasure (and not die)? If so, which one? Prove your answer using a truth table.

Group 2:

7. Zoey says, "Mel is a liar." Mel says, "Zoey and I are both truth-tellers." What is the identity of each person? Prove your answer first by a case analysis, and then by using a truth table.

Case analysis:

Case 1: Zoey is a truth-teller. Then Mel is a liar. Since Mel is a liar, her statement is false, and so at least one of the two is a liar. This is true since Mel is a liar, so this case works.

Case 2: Zoey is a liar. Then Mel is a truth-teller. This means that her statement is true, and so both Mel and Zoey are truth-tellers. But we already assumed that Zoey is a liar, so this cannot happen.

So the conclusion is that Zoey is a truth-teller and Mel is a liar.

Truth table:

Let p be "Zoey is a truth-teller" and q be "Mel is a truth-teller". Zoey's statement translates to $\neg q$. Mel's statement translates to $p \wedge q$. The truth table we get is:

p	q	Zoey says $\neg q$	Mel says $p \wedge q$
T	T	F / T	T / F
T	F	T / T	F / F
F	T	F / F	F / T
F	F	T / F	F / F

where in the 3rd and 4th columns, the first value comes from the values of p and q, while the second value comes from the identity of the person making the statement.

Only the second row is consistent, so it must be that Zoey is a truth-teller and Mel is a liar.

8. You are given three boxes. One of them contains gold, while the other two are empty. Box 1 says, "This box is empty." Box 2 says, "This box is empty." Box 3 says, "The gold is in Box 2." One of the messages is true, while the other two are false.

   Which box has the gold? Prove your answer using a truth table.

Group 3:

9. Zoey says, "of Mel and I, exactly one is a truth-teller." Mel says, "only a liar would say that Zoey is a liar." What is the identity of each person? Prove your answer first by a case analysis, and then by using a truth table.

Case analysis:

Before doing the case analysis, it might be helpful to decipher Mel's statement. She claims that a liar would say that Zoey is a liar – this statement is saying that it is false that Zoey is a liar, and so Zoey is a truth-teller. So from now on, we'll take Mel's statement to be "Zoey is a truth-teller".

Case 1: Zoey is a truth-teller. Then since her statement is true and she is a truth-teller, Mel must be a liar (there can only be one truth-teller). So Mel's statement is false, that is, Zoey is not a truth-teller. But we already assumed that Zoey is a truth-teller, so this cannot happen.

Case 2: Zoey is a liar. Then her statement is false, and so either both are truth-tellers, or both are liars. We already know Zoey is a liar, which would mean that Mel must be a liar too.

It's good to check this, so let's suppose that Mel is a liar. Then Mel's statement is false, that is, Zoey is a liar, which is indeed true. So this case works out.

So the conclusion is that both are liars.

Truth table:

Let p be "Zoey is a truth-teller" and q be "Mel is a truth-teller". Zoey's statement translates to $(p \wedge \neg q) \vee (\neg p \wedge q)$. Mel's statement translates to p. The truth table we get is:

		Zoey says	Mel says
p	q	$(p \wedge \neg q) \vee (\neg p \wedge q)$	p
T	T	F / T	T / T
T	F	T / T	T / F
F	T	T / F	F / T
F	F	F / F	F / F

where in the 3rd and 4th columns, the first value comes from the values of p and q, while the second value comes from the identity of the person making the statement.

Only the last row is consistent, so it must be that Zoey and Mel are both liars.

10. You are lost in a maze, when you come upon three possible roads. One of them is paved with gold, the second is paved with marble, and the last is paved with stone. A guard stands in front of each road. The guard of the gold road says: "This road leads to the exit. Moreover, if the stone road leads to the exit, then so does the marble road." The guard of the marble road says: "Neither the gold road nor the stone road lead to the exit." The guard of the stone road says: "Follow the gold road and you'll reach the exit; follow the marble road and you'll be lost."

    Which road should you take? Prove your answer using a truth table.

Challenge:

11. You've time-traveled back to medieval times, but the king sees you using your phone and accuses you of practicing magic, and throws you into the dungeon. However, the dungeon guard hates his job and doesn't care if you escape or die. In the dungeon, there are two doors. One leads out of the dungeon, but the other leads to the cage where they keep hungry lions for executions (and you really

don't feel like getting eaten by lions today). The guard is willing to answer exactly one question, but you don't know if he is a truth-teller or a liar. Can you figure out which door leads to freedom?