ESP Math 294

Week 3 Solutions

Spring 2021

Quantifiers

Date, 2021

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1. Write the following statement as a logical formula: For all real numbers x and y, if $x \neq y$, then $x^2 + y^2 > 0$. Is this statement true?

   > The formula should be $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [(x \neq y) \Rightarrow (x^2 + y^2 > 0)]$. The parentheses and brackets are not strictly necessary, but it helps to see how each sub-statement is separated out.
   >
   > The statement is <u>true</u>.

2. Translate the following logical formula into a plain English sentence, without using any variables:

   $$\forall a \in \mathbb{Q}, \forall b \in \mathbb{Q}, (a < b \Rightarrow \exists c \in \mathbb{Q}, [a < c < b]).$$

   Is this statement true?

   > The statement says that between any two rational numbers, there is another rational number.
   >
   > The statement is <u>true</u>.

3. If you were asked to prove a statement of the form

   $$\forall x \in X, \exists y \in Y, P(x, y),$$

   where $P(x, y)$ is a statement with variables x and y ranging over sets X and Y, what should be your process to prove this statement?

   > First, we would let $x \in X$. Then, we want to prove "$\exists y \in Y, P(x, y'$". To do this, we would have to find a specific element $a \in Y$ so that $P(x, a)$ is true.

4. If you were asked to prove a statement of the form

   $$(\exists x \in X, P(x)) \Rightarrow (\forall y \in Y, Q(y)),$$

   where $P(x)$ is a statement with variable x ranging over X and $Q(y)$ is a statement with variable y ranging over Y, what should be your process to prove this statement?

   > We would first assume the "if" part of the implication. That is, we would assume that there is an element a in the set X for which $P(x)$ is true.
   >
   > Then, we want to use that to prove the "then" part of the implication. That is, we want to prove "$\forall y \in Y, Q(y)$". In order to prove this, we would let y be in Y, and prove that $Q(y)$ is true.

5. We say that a proposition r is *stronger* than another proposition s if $r \Rightarrow s$.

   Let $P(x, y)$ be a statement with variables x and y ranging over sets X and Y. Consider the two propositions
   $$\forall x \in X, \exists y \in Y, P(x, y) \tag{1}$$
   and
   $$\exists y \in Y, \forall x \in X, P(x, y). \tag{2}$$
   Which of the two is stronger? Prove your answer.

   > Statement (2) is stronger.
   >
   > Assume that (2) is true. We want to prove (1).
   >
   > Let $a \in X$. The goal is to show $\exists y \in Y, P(a, y)$. Since (2) is true, we know that there is an element $b \in Y$ such that $\forall x \in X, P(x, b)$. Since $a \in X$ and $\forall x \in X, P(x, b)$, it follows that $P(a, b)$ is true. We have found an element $b \in Y$ so that $P(a, b)$ is true, which is exactly what our goal was.
   >
   > Therefore, (1) is true, as desired. □

6. Let $P(x)$ be a statement with variable x ranging over X.

   (a) Prove that $\neg(\exists x \in X, P(x))$ if and only if $\forall x \in X, \neg P(x)$. Recall that to prove $r \iff s$, you have to prove $r \to s$ and $s \to r$.

   (hint: if q is a statement and you want to prove $\neg q$, one strategy is to assume that q is true and derive something false. This indicates that q could not have been true to begin with and so q is false – this is known as a *proof by contradiction*)

   (b) Use part (a) to prove that $\neg(\forall x \in X, P(x))$ if and only if $\exists x \in X, \neg P(x)$.

   (hint: you should use the *law of double negation*, which says that if q is a statement, then $q \iff \neg\neg q$)

   (c) In words, what do parts (a) and (b) say? (these are sometimes known as *de Morgan's laws for quantifiers*)

7. (**Challenge**) Write out a logical formula that expresses the statement "there exists exactly one $x \in X$ such that $P(x)$ is true".

   This is sometimes written as $\exists! x \in X, P(x)$ (where the quantifier $\exists!$ is read as "exists unique"). Define a new quantifier $\forall!$ so that $\forall!$ satisfies de Morgan's laws for quantifiers (see Problem 6)