ESP Math 294

Week 3

Spring 2021

Quantifiers

Date, 2021

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1. Write the following statement as a logical formula: For all real numbers x and y, if $x \neq y$, then $x^2 + y^2 > 0$. Is this statement true?

2. Translate the following logical formula into a plain English sentence, without using any variables:

$$\forall a \in \mathbb{Q}, \forall b \in \mathbb{Q}, (a < b \Rightarrow \exists c \in \mathbb{Q}, [a < c < b]).$$

   Is this statement true?

3. If you were asked to prove a statement of the form

$$\forall x \in X, \exists y \in Y, P(x, y),$$

   where $P(x, y)$ is a statement with variables x and y ranging over sets X and Y, what should be your process to prove this statement?

4. If you were asked to prove a statement of the form

$$(\exists x \in X, P(x)) \Rightarrow (\forall y \in Y, Q(y)),$$

   where $P(x)$ is a statement with variable x ranging over X and $Q(y)$ is a statement with variable y ranging over Y, what should be your process to prove this statement?

5. We say that a proposition r is *stronger* than another proposition s if $r \Rightarrow s$.

   Let $P(x, y)$ be a statement with variables x and y ranging over sets X and Y. Consider the two propositions

$$\forall x \in X, \exists y \in Y, P(x, y)$$

   and

$$\exists y \in Y, \forall x \in X, P(x, y).$$

   Which of the two is stronger? Prove your answer.

6. Let $P(x)$ be a statement with variable x ranging over X.

   (a) Prove that $\neg(\exists x \in X, P(x))$ if and only if $\forall x \in X, \neg P(x)$. Recall that to prove $r \iff s$, you have to prove $r \rightarrow s$ and $s \rightarrow r$.

   (hint: if q is a statement and you want to prove $\neg q$, one strategy is to assume that q is true and derive something false. This indicates that q could not have been true to begin with and so q is false – this is known as a *proof by contradiction*)

   (b) Use part (a) to prove that $\neg(\forall x \in X, P(x))$ if and only if $\exists x \in X, \neg P(x)$.

   (hint: you should use the *law of double negation*, which says that if q is a statement, then $q \iff \neg\neg q$)

   (c) In words, what do parts (a) and (b) say? (these are sometimes known as *de Morgan's laws for quantifiers*)

7. (**Challenge**) Write out a logical formula that expresses the statement "there exists exactly one $x \in X$ such that $P(x)$ is true".

   This is sometimes written as $\exists! x \in X, P(x)$ (where the quantifier $\exists!$ is read as "exists unique"). Define a new quantifier $\forall!$ so that $\forall!$ satisfies de Morgan's laws for quantifiers (see Problem 6)