ESP Math 294

WEEK 4 SOLUTIONS
SETS I

Spring 2021

February 2, 2021

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1. Write the following sets using list or implied list notation:

   (a) $\{n \in \mathbb{Z} \mid n^2 < 20\}$

   Solution: $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

   (b) $\{3n + 1 \mid n \in \mathbb{Z}\}$

   Solution: $\{\dots, -5, -2, 1, 4, 7, \dots\}$

2. Write the following sets using set-builder notation:

   (a) $\{\dots, -4, -1, 2, 5, 8, \dots\}$

   Solution: $\{3n - 1 \mid n \in \mathbb{Z}\}$. Other solutions are possible, e.g. $3n - 4$ or $3n + 2$.

   (b) $\{-3, -2, -1, 0, 1, 2, 3\}$

   Solution: $\{n \in \mathbb{Z} \mid -3 \leq n \leq 3\}$. Other solutions are also possible.

3. Let X and Y be sets, whose elements come from a universal set $\mathcal{U}$. Prove, via a double containment proof, that $X \setminus Y = X \cap Y^c$.

   Solution:

   Proof. First, we prove that $X \setminus Y \subseteq X \cap Y^c$. Let $a \in X \setminus Y$. The goal is to prove that $a \in X \cap Y^c$, that is, we need to prove $a \in X$ (1) and $a \in Y^c$ (2). By definition of relative complement, this means that $a \in X$ (which shows (1) is true) and $a \notin Y$. By definition of complement, $a \notin Y$ means that $a \in Y^c$. So both (1) and (2) are true and $a \in X \cap Y^c$, as desired.

   Next, we prove that $X \cap Y^c \subseteq X \setminus Y$. Let $a \in X \cap Y^c$. The goal is to prove that $a \in X \setminus Y$, that is, we need to prove $a \in X$ (3) and $a \notin Y$ (4). By the definition of intersection, we know $a \in X$ (which shows (3) is true) and $a \in Y^c$. By definition of complement, $a \in Y^c$ means that $a \notin Y$. So both (3) and (4) are true and $a \in X \setminus Y$, as desired. □

   Note: I labeled some of the statements I was trying to prove to make the proof easier to read, but you don't necessarily have to do this. You also don't necessarily have to write out every single "by definition of..." statement – it makes it more clear why each step follows from the previous, but it can also make the proof very long and wordy.

We mentioned that the naive "definition" of a set as any collection of objects causes issues. In this problem, we'll explore one such issue, known as *Russell's paradox*.

4. Let $R = \{X \mid X \text{ is a set, and } X \notin X\}$ (remember that sets themselves can be considered as "objects", so a set can be an element of another set). In words, R is the set of all sets that don't contain themselves. Notice that for any object a and any set B, exactly one of $a \in B$ and $a \notin B$ is true.

   (a) Consider the case where $R \in R$. What does this tell you about R?

   Solution: Since $R \in R$, the definition of R says that R is a set and $R \notin R$. But this is a contradiction, since we can't have both $R \in R$ and $R \notin R$.

   (b) Consider the case where $R \notin R$. What does this tell you about R?

   Solution: Since $R \notin R$ and R is a set, R fits the condition in the definition of R, and so $R \in R$. But this is a contradiction!

(c) Based on your answers for parts (a) and (b), what is the issue with our "set" R?

Solution: (a) and (b) cover all possibilities, and in both cases we reached a contradiction. So it doesn't make logical sense for R to actually be a set.

(Note: mathematicians have devised more clever definitions for sets that get around things like Russell's paradox, but they fall well out of the scope of this course.)

For the following problems, we will need a definition:

Definition. The *symmetric difference* of two sets X and Y is the set $X \triangle Y$ given by

$$X \triangle Y := \{a \mid a \in X \text{ or } a \in Y, \text{ but not both}\}.$$

5. What is the symmetric difference $A \triangle B$, where $A = \{1, 2, 3, 4, 5, a, b, c\}$ and $B = \{2, 3, 5, 6, c, d\}$?
6. Let X and Y be sets. Prove that $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.
7. Let X be a set. Prove that $X \triangle X = \varnothing$ and $X \triangle \varnothing = X$.
8. Let X and Y be sets. Prove that $X = Y$ if and only if $X \triangle Y = \varnothing$.

For the following problems, we will need some definitions.

We say that two sets A and B are *disjoint* if $A \cap B = \varnothing$ (in other words, A and B don't intersect).

Definition. Let X be a (nonempty) set, and let $\mathcal{C}$ be a set of subsets of X. We say that $\mathcal{C}$ is *pairwise disjoint* if for every $A, B \in \mathcal{C}$ with A and B not equal, A and B are disjoint (note that A and B are both subsets of X).

Let $X = \{0, 1, 2, 3, 4, 5\}$.

9. Let $A = \{1, 2, 3\}$, $B = \{0, 4\}$, and $C = \{4, 5\}$. Is the collection $\mathcal{C} = \{A, B, C\}$ pairwise disjoint? Justify your answer.
10. Let $D = \{1, 3\}$, $E = \{0, 2, 4\}$, and $F = \{5\}$. Is the collection $\mathcal{C} = \{D, E, F\}$ pairwise disjoint? Justify your answer.

Definition. Let X be a (nonempty) set, and let $\mathcal{C}$ be a set of subsets of X. We say that $\mathcal{C}$ is a *partition* of X if the empty set is <u>not</u> an element of $\mathcal{C}$, $\mathcal{C}$ is pairwise disjoint, and every element of X belongs to a subset from $\mathcal{C}$.

11. Let X be a (nonempty) set and let $\mathcal{C}$ be a set of (nonempty) subsets of X. Consider the following three conditions:

    (i) The set $\mathcal{C}$ is a partition of X (that is, it satisfies Definition).

    (ii) The empty set is <u>not</u> an element of $\mathcal{C}$, and every element of X belongs to exactly one subset from $\mathcal{C}$.

    (iii) The set $\mathcal{C}$ satisfies the following three properties:

    (1) the empty set is <u>not</u> an element of $\mathcal{C}$;

    (2) for every $A, B \in \mathcal{C}$, either $A = B$ or A and B are disjoint; and

    (3) the union of all of the sets in $\mathcal{C}$ is X.

    We will prove that the three conditions (i) – (iii) are all equivalent, which means that each condition can serve as the definition for a partition, in the following way:

    (a) Prove that if $\mathcal{C}$ satisfies (i), then $\mathcal{C}$ also satisfies (ii).

(b) Prove that if $\mathcal{C}$ satisfies (ii), then $\mathcal{C}$ also satisfies (iii).

(c) Prove that if $\mathcal{C}$ satisfies (iii), then $\mathcal{C}$ also satisfies (i).