Week 4
Sets I

February 2, 2021

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1. Write the following sets using list or implied list notation:
   (a) $\{n \in \mathbb{Z} \mid n^2 < 20\}$
   (b) $\{3n + 1 \mid n \in \mathbb{Z}\}$
2. Write the following sets using set-builder notation:
   (a) $\{\ldots, -4, -1, 2, 5, 8, \ldots\}$
   (b) $\{-3, -2, -1, 0, 1, 2, 3\}$
3. Let X and Y be sets, whose elements come from a universal set $\mathcal{U}$. Prove, via a double containment proof, that $X \setminus Y = X \cap Y^c$.

We mentioned that the naive "definition" of a set as any collection of objects causes issues. In this problem, we'll explore one such issue, known as *Russell's paradox*.

4. Let $R = \{X \mid X \text{ is a set, and } X \notin X\}$ (remember that sets themselves can be considered as "objects", so a set can be an element of another set). In words, R is the set of all sets that don't contain themselves. Notice that for any object a and any set B, exactly one of $a \in B$ and $a \notin B$ is true.
   (a) Consider the case where $R \in R$. What does this tell you about R?
   (b) Consider the case where $R \notin R$. What does this tell you about R?
   (c) Based on your answers for parts (a) and (b), what is the issue with our "set" R?

(Note: mathematicians have devised more clever definitions for sets that get around things like Russell's paradox, but they fall well out of the scope of this course.)

For the following problems, we will need a definition:

Definition. The *symmetric difference* of two sets X and Y is the set $X \triangle Y$ given by

$$X \triangle Y := \{a \mid a \in X \text{ or } a \in Y, \text{ but not both}\}.$$

5. What is the symmetric difference $A \triangle B$, where $A = \{1, 2, 3, 4, 5, a, b, c\}$ and $B = \{2, 3, 5, 6, c, d\}$?
6. Let X and Y be sets. Prove that $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.
7. Let X be a set. Prove that $X \triangle X = \varnothing$ and $X \triangle \varnothing = X$.
8. Let X and Y be sets. Prove that $X = Y$ if and only if $X \triangle Y = \varnothing$.

For the following problems, we will need some definitions.

We say that two sets A and B are *disjoint* if $A \cap B = \varnothing$ (in other words, A and B don't intersect).

Definition. Let X be a (nonempty) set, and let $\mathcal{C}$ be a set of subsets of X. We say that $\mathcal{C}$ is *pairwise disjoint* if for every $A, B \in \mathcal{C}$ with A and B not equal, A and B are disjoint (note that A and B are both subsets of X).

Let $X = \{0, 1, 2, 3, 4, 5\}$.

9. Let $A = \{1, 2, 3\}$, $B = \{0, 4\}$, and $C = \{4, 5\}$. Is the collection $\mathcal{C} = \{A, B, C\}$ pairwise disjoint? Justify your answer.

10. Let $D = \{1, 3\}$, $E = \{0, 2, 4\}$, and $F = \{5\}$. Is the collection $\mathcal{C} = \{D, E, F\}$ pairwise disjoint? Justify your answer.

Definition. Let X be a (nonempty) set, and let $\mathcal{C}$ be a set of subsets of X. We say that $\mathcal{C}$ is a *partition* of X if the empty set is <u>not</u> an element of $\mathcal{C}$, $\mathcal{C}$ is pairwise disjoint, and every element of X belongs to a subset from $\mathcal{C}$.

11. Let X be a (nonempty) set and let $\mathcal{C}$ be a set of (nonempty) subsets of X. Consider the following three conditions:

    (i) The set $\mathcal{C}$ is a partition of X (that is, it satisfies Definition).

    (ii) The empty set is <u>not</u> an element of $\mathcal{C}$, and every element of X belongs to exactly one subset from $\mathcal{C}$.

    (iii) The set $\mathcal{C}$ satisfies the following three properties:

    (1) the empty set is <u>not</u> an element of $\mathcal{C}$;

    (2) for every $A, B \in \mathcal{C}$, either $A = B$ or A and B are disjoint; and

    (3) the union of all of the sets in $\mathcal{C}$ is X.

    We will prove that the three conditions (i) – (iii) are all equivalent, which means that each condition can serve as the definition for a partition, in the following way:

    (a) Prove that if $\mathcal{C}$ satisfies (i), then $\mathcal{C}$ also satisfies (ii).

    (b) Prove that if $\mathcal{C}$ satisfies (ii), then $\mathcal{C}$ also satisfies (iii).

    (c) Prove that if $\mathcal{C}$ satisfies (iii), then $\mathcal{C}$ also satisfies (i).