ESP Math 294 | **Week 5 Solutions** | Spring 2021

Week 5 Solutions
Sets II

February 9, 2021

Power set

1. Write out the elements of $\mathcal{P}(\{0,1,2\})$.

   Solution: $\mathcal{P}(\{0,1,2\}) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$.

2. Write out the elements of $\mathcal{P}(\varnothing), \mathcal{P}(\mathcal{P}(\varnothing))$, and $\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing)))$.

   Solution: $\mathcal{P}(\varnothing) = \{\varnothing\}$.

   $\mathcal{P}(\mathcal{P}(\varnothing)) = \{\varnothing, \{\varnothing\}\}$.

   $\mathcal{P}(\mathcal{P}(\varnothing)) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$.

3. (a) Prove that $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$ implies $X \subseteq Y$.

   Solution 1:

   Proof. Assume that $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$. We want to prove that $X \subseteq Y$.

   Let $a \in X$. The goal is to prove that $a \in Y$.

   Consider the set $\{a\}$. Since $a \in X$, it follows that $\{a\} \subseteq X$. By the definition of the power set, $\{a\} \in \mathcal{P}(X)$. Then since $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$, we have that $\{a\} \in \mathcal{P}(Y)$. The definition of the power set tells us that $\{a\} \subseteq Y$. Hence, we have that $a \in Y$, as desired. □

   Solution 2:

   Proof. Assume that $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$. We want to prove that $X \subseteq Y$.

   Recall that we proved in class that $X \in \mathcal{P}(X)$ for any set X. Since $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$, we have that $X \in \mathcal{P}(Y)$. By the definition of the power set, $X \subseteq Y$, as desired. □

   (b) Prove that if $X \subsetneq Y$ (that is, $X \subseteq Y$ but $X \neq Y$), then $\mathcal{P}(X) \neq \mathcal{P}(Y)$ (hint: try to come up with an element of $\mathcal{P}(Y)$ that is not an element of $\mathcal{P}(X)$).

   Solution:

   Proof. In order to prove that $\mathcal{P}(X) \neq \mathcal{P}(Y)$, we want to give an element of $\mathcal{P}(Y)$ that is not in $\mathcal{P}(X)$.

   Since $X \subsetneq Y$, there is an element a such that $a \in Y$ and $a \notin X$. Notice that $\{a\} \subseteq Y$ (since $a \in Y$), but $\{a\} \nsubseteq X$ (since $a \notin X$). By the definition of power set, $\{a\} \in \mathcal{P}(Y)$, but $\{a\} \notin \mathcal{P}(X)$. So $\mathcal{P}(X) \neq \mathcal{P}(Y)$. □

4. For each of the following statements, determine if the statement is true for all sets X and Y, false for all sets X and Y, or true for some choices of X and Y and false for others. Justify your answers!

   (a) $\mathcal{P}(X \cup Y) = \mathcal{P}(X) \cup \mathcal{P}(Y)$

   (b) $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$

   (c) $\mathcal{P}(X \setminus Y) = \mathcal{P}(X) \setminus \mathcal{P}(Y)$

   (d) $\mathcal{P}(X \times Y) = \mathcal{P}(X) \times \mathcal{P}(Y)$

5. **(Challenge)** Let X be a set that contains exactly n elements (we say that X has *cardinality* n and denote this as $|X| = n$). How many elements does $\mathcal{P}(X)$ have? Does this depend on what X is?

Indexed families of sets

6. For a real number r, define S_r to be the interval $[r-1, r+2]$. Let $A = \{1, 3, 4\}$. Determine $\bigcup_{i \in A} S_i$ and $\bigcap_{i \in A} S_i$.

   Solution: Notice that $A_1 = [0, 3], A_3 = [2, 5]$, and $A_4 = [3, 6]$.

   Then $\bigcup_{i \in A} S_i = [0, 6]$ and $\bigcap_{i \in A} S_i = \{3\}$.

7. For each $n \geq 1$, let $X_n = [0, 1 + \frac{1}{n})$ as in the example we did earlier. Write $\bigcup_{n \geq 1} \left[0, 1 + \frac{1}{n}\right)$ as an interval.

   Solution: $\bigcup_{n \geq 1} \left[0, 1 + \frac{1}{n}\right) = \left[0, \frac{3}{2}\right)$.

8. Find an indexed family of sets $\{X_n \mid n \in \mathbb{N}\}$ such that all three of the following hold (it may be helpful to first figure out, in words, what each of the conditions mean):

   (a) $\displaystyle\bigcup_{n \in \mathbb{N}} X_n = \mathbb{N}$;

   (b) $\displaystyle\bigcap_{n \in \mathbb{N}} X_n = \mathbb{N}$; and

   (c) For any $i, j \in \mathbb{N}$, $X_i \cap X_j \neq \varnothing$.

9. **(Challenge)** For this problem, we will need the following definition:

   Definition. We say that a subset $U \subseteq R$ is *open* if, for all $a \in U$, there is a number $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq U$.

   As an example, the interval $(0, 1)$ is open (if $0 < a < 1$, let δ be the minimum of $\frac{a}{2}$ and $\frac{1-a}{2}$. Verify that $(a - \delta, a + \delta) \subseteq (0, 1)$). On the other hand, the interval $[0, 1]$ is <u>not</u> open (let $a = 1$, and verify that no matter what $\delta > 0$ you choose, $(a - \delta, a + \delta) \not\subseteq [0, 1]$).

   In this problem, we will show that an intersection of finitely many open sets is open, but that an intersection of infinitely many open sets might not be open.

   (a) Let $n \geq 1$ and suppose that $U_1, \ldots, U_n$ are all open subsets of $\mathbb{R}$. Prove that the intersection $U_1 \cap \cdots \cap U_n$ is open.

   (b) Prove that $(0, 1 + \frac{1}{n})$ is open for all $n \geq 1$, but that $\bigcap_{n \geq 1} (0, 1 + \frac{1}{n})$ is not open.