ESP Math 294 | **Week 5** | Spring 2021

Sets II

February 9, 2021

Power set

1. Write out the elements of $\mathcal{P}(\{0,1,2\})$.
2. Write out the elements of $\mathcal{P}(\varnothing), \mathcal{P}(\mathcal{P}(\varnothing))$, and $\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing)))$.
3. (a) Prove that $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$ implies $X \subseteq Y$.

   (b) Prove that if $X \subsetneq Y$ (that is, $X \subseteq Y$ but $X \neq Y$), then $\mathcal{P}(X) \neq \mathcal{P}(Y)$ (hint: try to come up with an element of $\mathcal{P}(Y)$ that is not an element of $\mathcal{P}(X)$).
4. For each of the following statements, determine if the statement is true for all sets X and Y, false for all sets X and Y, or true for some choices of X and Y and false for others. Justify your answers!

   (a) $\mathcal{P}(X \cup Y) = \mathcal{P}(X) \cup \mathcal{P}(Y)$

   (b) $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$

   (c) $\mathcal{P}(X \setminus Y) = \mathcal{P}(X) \setminus \mathcal{P}(Y)$

   (d) $\mathcal{P}(X \times Y) = \mathcal{P}(X) \times \mathcal{P}(Y)$
5. **(Challenge)** Let X be a set that contains exactly n elements (we say that X has *cardinality* n and denote this as $|X| = n$). How many elements does $\mathcal{P}(X)$ have? Does this depend on what X is?

Indexed families of sets

6. For a real number r, define S_r to be the interval $[r-1, r+2]$. Let $A = \{1,3,4\}$. Determine $\bigcup_{i \in A} S_i$ and $\bigcap_{i \in A} S_i$.
7. For each $n \geq 1$, let $X_n = [0, 1+\frac{1}{n})$ as in the example we did earlier. Write $\bigcup_{n \geq 1} \left[0, 1+\frac{1}{n}\right)$ as an interval.
8. Find an indexed family of sets $\{X_n \mid n \in \mathbb{N}\}$ such that all three of the following hold (it may be helpful to first figure out, in words, what each of the conditions mean):

   (a) $\displaystyle\bigcup_{n \in \mathbb{N}} X_n = \mathbb{N}$;

   (b) $\displaystyle\bigcap_{n \in \mathbb{N}} X_n = \mathbb{N}$; and

   (c) For any $i, j \in \mathbb{N}$, $X_i \cap X_j \neq \varnothing$.
9. **(Challenge)** For this problem, we will need the following definition:

   Definition. We say that a subset $U \subseteq R$ is *open* if, for all $a \in U$, there is a number $\delta > 0$ such that $(a-\delta, a+\delta) \subseteq U$.

   As an example, the interval $(0,1)$ is open (if $0 < a < 1$, let δ be the minimum of $\frac{a}{2}$ and $\frac{1-a}{2}$. Verify that $(a-\delta, a+\delta) \subseteq (0,1)$). On the other hand, the interval $[0,1]$ is <u>not</u> open (let $a = 1$, and verify that no matter what $\delta > 0$ you choose, $(a-\delta, a+\delta) \not\subseteq [0,1]$).

In this problem, we will show that an intersection of finitely many open sets is open, but that an intersection of infinitely many open sets might not be open.

(a) Let $n \geq 1$ and suppose that $U_1, \ldots, U_n$ are all open subsets of $\mathbb{R}$. Prove that the intersection $U_1 \cap \cdots \cap U_n$ is open.

(b) Prove that $(0, 1 + \frac{1}{n})$ is open for all $n \geq 1$, but that $\bigcap_{n \geq 1}(0, 1 + \frac{1}{n})$ is not open.