ESP Math 294 | Spring 2021

Week 6 Solutions
Contrapositive & Contradiction

February 17, 2021

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1. Use a proof by contrapositive to show that if n^2 is even, then n is even.

Solution:

Proof. We do a proof by contrapositive–the contrapositive is "if n is odd, then n^2 is even."

Assume that n is odd. The goal is to prove that n^2 is odd, that is, to prove that there is some integer l such that $n^2 = 2l + 1$.

Since n is odd, there is an integer k such that $n = 2k + 1$. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2) + 1$. Since $2k^2 + 2$ is an integer, we conclude that n^2 is odd, as desired. □

2. Let $x, y \in \mathbb{R}$. Use a proof by contrapositive to show that if $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$.

Solution:

Proof. We do a proof by contrapositive–the contrapositive is "if $y > x$, then $y^3 + yx^2 > x^3 + xy^2$."

Assume that $y > x$. First, since the square of any real number is nonnegative, $x^2 \geq 0$ and $y^2 \geq 0$. So $x^2 + y^2 \geq 0$. Also, notice that in order for $x^2 + y^2 = 0$, we would need to have that $x = y = 0$. However, since $y > x$, this cannot happen, so actually $x^2 + y^2 > 0$.

Next, we have that

$$\begin{aligned} y > x &\Rightarrow y(x^2 + y^2) > x(x^2 + y^2) && \text{valid because } x^2 + y^2 > 0 \\ &\Rightarrow yx^2 + y^3 > x^3 + xy^2 \\ &\Rightarrow y^3 + yx^2 > x^3 + xy^2 \end{aligned}$$

as desired. □

3. Use a proof by contradiction to show that $\sqrt{3}$ is irrational. You may use the fact that for any integer n, if n^2 is divisible by 3, then n is divisible by 3.

Solution:

Proof. Assume for the sake of contradiction that $\sqrt{3}$ is rational. This means that there are integers p and q such that $\sqrt{3} = \frac{p}{q}$. In particular, we can choose p and q so that they have no common factors ($\frac{p}{q}$ is fully simplified).

Then

$$\begin{aligned} \sqrt{3} = \frac{p}{q} &\Rightarrow 3 = \frac{p^2}{q^2} \\ &\Rightarrow 3q^2 = p^2 \end{aligned}$$

which means that p^2 is divisible by 3. Thus p is also divisible by 3, and so there is an integer k such that $p = 3k$.

$$\begin{aligned} &\Rightarrow 3q^2 = (3k)^2 = 9k^2 \\ &\Rightarrow q^2 = 3k^2. \end{aligned}$$

Therefore, q^2 is divisible by 3, and so q is divisible by 3.

But now we have concluded that p and q are both divisible by 3, which contradicts the fact that p and q have no common factors!

So $\sqrt{3}$ is actually irrational. $\square$

4. Let $a, b \in \mathbb{R}$, where a is rational and b is irrational. Use a proof by contradiction to show that ab is irrational.

Solution:

Proof. Assume for the sake of contradiction that ab is rational. This means that there are integers p and q such that $ab = \frac{p}{q}$.

We also know that a is rational, so there are integers m and n such that $a = \frac{m}{n}$.

Then

$$b = \frac{ab}{a} = \frac{p}{q} \div \frac{m}{n} = \frac{p}{q} \cdot \frac{n}{m} = \frac{pn}{qm}.$$

This would mean that b is rational, but we assumed that b was irrational. This is a contradiction!

So ab is irrational. $\square$

5. Give an example of a statement (doesn't have to be a mathematical statement) of the form $P \Rightarrow Q$ that is true, but the converse is false.

6. Let $a, b, c \in \mathbb{Z}$. Prove that if $a^2 + b^2 = c^2$, then abc is even. (You should decide whether you want to use a direct proof, contrapositive, or contradiction)

7. Let x and y be positive real numbers. Prove that $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$. (You should decide whether you want to use a direct proof, contrapositive, or contradiction)

8. Suppose that r_1 and r_2 are two different roots of the quadratic function $f(x) = x^2 + bx + c$ (recall that r is a a root of a polynomial $p(x)$ if $p(r) = 0$). Prove that $r_1 + r_2 = -b$ and $r_1 r_2 = c$. (You should decide whether you want to use a direct proof, contrapositive, or contradiction)

9. **(Challenge)** You've been selected to take part in a new game show. The first game involves you and two other players. Each player has either a red hat or a blue hat placed on their head. There are no mirrors, so you can't see the color of your own hat, but each player can see the other players' hats. Once the hats are placed, each player is supposed to raise their hand if they see at least one red hat. The first player to correctly guess their color wins. Guessing is not allowed, so players must justify their guess in order to win.

   When the hats are placed, you see two red hats, so you raise your hand. The other players also both raise their hands. After a few minutes, no one has made any guesses. What color should you guess for your own hat, and what justification should you give?