ESP Math 294 | Spring 2021

Week 8 Solutions
Induction II

March 2, 2021

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1. Prove that for all $n \in \mathbb{N}$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

2. Let $x \in \mathbb{R}$ with $x > -1$. Prove that for all $n \in \mathbb{N}$, $1 + nx \leq (1+x)^n$. (Hint: at some point, you might want to consider two cases, one where $x \geq 0$, and one where $-1 < x < 0$.)

   See last week's solutions for Problems 1 and 2.

3. Prove that for all $n \in \mathbb{N}$, $\frac{d^n}{dx^n}(xe^x) = (n+x)e^x$. (Hint: use the product rule.)

   Solution:

   Proof. We proceed by induction on n. $P(n)$ is the statement $\frac{d^n}{dx^n}(xe^x) = (n+x)e^x$.

   Base case: We want to prove $P(0)$, that is, $\frac{d^0}{dx^0}(xe^x) = (0+x)e^x$. The left hand side is simply xe^x, while the right hand side is also xe^x, so $P(0)$ is true.

   Induction step: Let $n \in \mathbb{N}$, and assume that $P(n)$ holds–that is, that $\frac{d^n}{dx^n}(xe^x) = (n+x)e^x$.

   We want to prove $P(n+1)$, that is, that $\frac{d^{n+1}}{dx^{n+1}}(xe^x) = ((n+1)+x)e^x$.

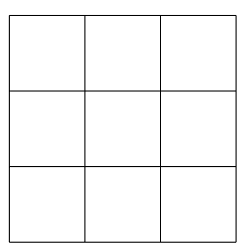
   Starting from the right hand side, we calculate:

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}(xe^x) &= \frac{d}{dx}\left(\frac{d^n}{dx^n}(xe^x)\right) \\
&= \frac{d}{dx}\left((n+x)e^x\right) \\
&= (n+x)e^x + e^x && \text{by the product rule} \\
&= ((n+1)+x)e^x
\end{aligned}$$

   exactly as desired.

   So by induction, the statement is true for all $n \in \mathbb{N}$. □

4. Consider an $n \times n$ grid of squares. For example, here's a picture of a 3×3 grid:



   Within an $n \times n$ grid, how many squares can you find? For example, in the 3×3 case, there are nine 1×1 squares, four 2×2 squares, and one 3×3 square, so there are 14 squares total.

   Write your answer as a summation, and use induction to prove that this summation is $\frac{n(n+1)(2n+1)}{6}$.

Solution:

The number of squares in an $n \times n$ grid is $\sum_{i=1}^{n} i^2$. Why is this true? First, notice that in an $n \times n$ grid, there are n^2 many 1×1 squares.

Next, we try to count the number of 2×2 squares. Consider the 2×2 square in the top left corner of the grid. We can shift it over to the right up to $n-1$ many times, and we can also shift it downwards up to $n-1$ times. Every possible 2×2 square is obtained by shifting this top-left square to the right and downwards a certain number of times. So there are $(n-1)^2$ many of the 2×2 squares.

Another way of thinking about this is that the 2×2 squares form a grid of their own that is $n-1$ across by $n-1$ down. So there are $(n-1)^2$ many of these squares.

This reasoning applies for any size of square – to count the number of $k \times k$ squares, where $1 \leq k \leq n$, we can see that the $k \times k$ squares form a grid that is $n-k+1$ across by $n-k+1$ down. So there are n^2 1×1 squares, $(n-1)^2$ 2×2 squares, $(n-2)^2$ 3×3 squares, and so on until 1 $n \times n$ square. In total, this gives $1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \sum_{i=1}^{n} i^2$ squares in total.

We can now prove that the total number of squares (which is $\sum_{i=1}^{n} i^2$) is $\frac{n(n+1)(2n+1)}{6}$, using induction.

$P(n)$ is the statement $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Base case: We want to prove $P(1)$. This is the statement $\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2(1)+1)}{6}$. The left hand side is 1, and we can calculate the right hand side to see that it is also 1. So $P(1)$ is true.

Induction step: Let $n \geq 1$, and assume that $P(n)$ is true; that is, that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true.

We want to prove $P(n+1)$; that is, $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$.

Starting from the left hand side, we calculate:

$$
\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \left(\sum_{i=1}^{n} i^2\right) + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 && \text{by the induction hypothesis} \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \\
&= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)(n+2)(2(n+1)+1)}{6}
\end{aligned}
$$

exactly as desired.

So by induction, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 1$, and since the left hand side is the number of squares we can find in an $n \times n$ grid, the right hand side also gives the formula for the number of squares. □