ESP Math 294 | Spring 2021

Week 8
Induction II

March 2, 2021

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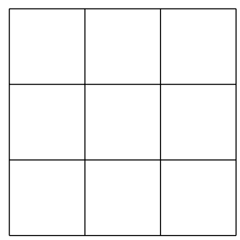
1. Prove that for all $n \in \mathbb{N}$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

2. Let $x \in \mathbb{R}$ with $x > -1$. Prove that for all $n \in \mathbb{N}$, $1 + nx \leq (1+x)^n$. (Hint: at some point, you might want to consider two cases, one where $x \geq 0$, and one where $-1 < x < 0$.)

3. Prove that for all $n \in \mathbb{N}$, $\frac{d^n}{dx^n}(xe^x) = (n+x)e^x$. (Hint: use the product rule.)

4. Consider an $n \times n$ grid of squares. For example, here's a picture of a 3×3 grid:

   Within an $n \times n$ grid, how many squares can you find? For example, in the 3×3 case, there are nine 1×1 squares, four 2×2 squares, and one 3×3 square, so there are 14 squares total.

   Write your answer as a summation, and use induction to prove that this summation is $\frac{n(n+1)(2n+1)}{6}$.

5. **(Challenge)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

   (a) Prove that there is a real number a such that $f(n) = an$ for all $n \in \mathbb{N}$.

   (b) Deduce that $f(n) = an$ for all $n \in \mathbb{Z}$.

   (c) Deduce that $f(x) = ax$ for all $x \in \mathbb{Q}$.

6. **(Challenge)** Read about the Principle of Strong Induction in the notes. Then, use strong induction to prove the following statement:

   Let P be a set of integers, all at least 2, with the following two properties:

   (a) $2 \in P$ and $3 \in P$;

   (b) If n is an integer with $n \geq 2$, then either $n \in P$, or there are elements $a, b \in P$ such that $n = ab$.

   Then any integer n with $n \geq 2$ is either an element of P, or can be written as a product of elements of P (that is, there are elements $p_1, \ldots, p_m \in P$, possibly with repeated numbers, such that $n = p_1 \cdots p_m$).