ESP Math 294 | **Week 9 Solutions** | Spring 2021

Week 9 Solutions
Functions

March 9, 2021

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1. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 2x$. What is $g[\mathbb{Z}]$, that is, the image of $\mathbb{Z}$ under g?

   Solution:

   The image of $\mathbb{Z}$ under g is the set of even integers.

   If you wanted to write a formal proof of this, it might look something like this:

   Proof. Let E be the set of even integers. We prove that $g[\mathbb{Z}] = E$ by double containment.

   ($g[\mathbb{Z}] \subseteq E$) Let $y \in g[\mathbb{Z}]$. Then $y = 2x$ for some $x \in \mathbb{Z}$ (by definition of image). We know that 2 times any integer is an even integer, so $y \in E$. Thus $g[\mathbb{Z}] \subseteq E$.

   ($E \subseteq g[\mathbb{Z}]$) Let $y \in E$ (that is, y is even). Every even integer is divisible by two, which means that there is an integer x so that $y/2 = x$. Then $g(x) = 2x = 2(y/2) = y$. So since there is an $x \in \mathbb{Z}$ such that $g(x) = y$, we conclude that $y \in g[\mathbb{Z}]$. Thus $E \subseteq g[\mathbb{Z}]$.

   Therefore, by double containment, $g[\mathbb{Z}] = E$ (that is, $g[\mathbb{Z}]$ is the set of all even integers). □

2. Let $f : X \to Y$ be any function, and let $U, V \subseteq X$. We proved that it is always true that $f[U \cap V] \subseteq f[U] \cap f[V]$.

   (a) Is it always true that $f[U] \cap f[V] \subseteq f[U \cap V]$? Prove your answer.

   (A useful reminder: in order to prove that it's always true, you should prove the statement only using the fact that f is a function and that $U, V \subseteq X$. But to prove that it's not always true, it's enough to give a single example of a function f and subsets $U, V \subseteq X$ for which the statement doesn't happen. This applies for all the other parts of this problem, as well as other problems on this worksheet.)

   Solution:

   This is **not** always true.

   Here is a counterexample:

   Let $X = \{1, 2\}$ and $Y = \{1\}$. Let $U = \{1\}$ and $V = \{2\}$ (notice that U and V are both subsets of X). Define $f : X \to Y$ by $f(1) = 1$ and $f(2) = 1$.

   We see that $f[U] = \{1\}$ (since $f(1) = 1$) and $f[V] = \{1\}$ (since $f(2) = 1$). So $f[U] \cap f[V] = \{1\}$.

   However, $U \cap V = \varnothing$. Then $f[U \cap V] = f[\varnothing] = \varnothing$.

   Clearly, $\{1\} \not\subseteq \varnothing$, so it's not true in this example that $f[U] \cap f[V] \subseteq f[U \cap V]$.

   (b) Is it always true that $f[U \cup V] \subseteq f[U] \cup f[V]$? Prove your answer.

   Solution: This is always true.

   Proof. Let $y \in f[U \cup V]$. By definition of image, there is some $x \in U \cup V$ such that $y = f(x)$.

   We consider the two cases: either $x \in U$ or $x \in V$.

   1) If $x \in U$, then since $y = f(x)$, $y \in f[U]$.
   2) On the other hand, if $x \in V$, then $y \in f[V]$.

   In either case, at least one of $y \in f[U]$ or $y \in f[V]$ is true, and so $y \in f[U] \cup f[V]$.

   So $f[U \cup V] \subseteq f[U] \cup f[V]$. □

(c) Is it always true that $f[U] \cup f[V] \subseteq f[U \cup V]$? Prove your answer.

Solution: This is always true.

Proof. Let $y \in f[U] \cup f[V]$. Then at least one of $y \in f[U]$ or $y \in f[V]$ is true.

We consider two cases:

1) If $y \in f[U]$, then there is $x \in U$ such that $y = f(x)$. Notice that $x \in U \cup V$, so $y \in f[U \cup V]$ as well.

2) On the other hand, if $y \in f[V]$ then there is $x \in V$ such that $y = f(x)$. Notice that $x \in U \cup V$, so $y \in f[U \cup V]$ as well.

In both cases, $y \in f[U \cup V]$, so $f[U] \cup f[V] \subseteq f[U \cup V]$. $\square$

3. Let $h : \mathbb{R} \to \mathbb{R}$ be given by $h(x) = |x|$ (remember that $|x|$ is the *absolute value of* x, which keeps positive numbers the same but turns negative numbers into positive numbers), and recall that (a, b) is the open interval between a and b, that is, it is all real numbers y so that $a < y < b$.

What is $h^{-1}[(-1, 5)]$? For any two real numbers a and b where $a < b$, what is $h^{-1}[(a, b)]$?

Solution:

$h^{-1}[(-1, 5)] = [-5, 5]$.

For the second part, we have three cases:

- Case 1: $a < b < 0$. Then $h^{-1}[(a, b)] = \varnothing$.
- Case 2: $a < 0$ and $b \geq 0$. Then $h^{-1}[(a, b)] = (-b, b)$.
- Case 3: $a \geq 0$ and $b \geq 0$. Then $h^{-1}[(a, b)] = (-b, -a) \cup (a, b)$.

4. Let $f : X \to Y$ be any function. For each of the following statements, determine whether or not the statement is true or false and provide a proof of your answer.

(a) For all $U, V \subseteq Y$, $f^{-1}[U \cap V] \subseteq f^{-1}[U] \cap f^{-1}[V]$.

Solution: This statement is always true.

Proof. Let $x \in f^{-1}[U \cap V]$. Then $f(x) \in U \cap V$ (by definition of preimage). So $f(x) \in U$ and $f(x) \in V$. Again by definition of preimage, $x \in f^{-1}[U]$ and $x \in f^{-1}[V]$. Therefore, $x \in f^{-1}[U] \cap f^{-1}[V]$. $\square$

(b) For all $U, V \subseteq Y$, $f^{-1}[U] \cap f^{-1}[V] \subseteq f^{-1}[U \cap V]$.

(c) For all $U, V \subseteq Y$, $f^{-1}[U \cup V] \subseteq f^{-1}[U] \cup f^{-1}[V]$.

(d) For all $U, V \subseteq Y$, $f^{-1}[U] \cup f^{-1}[V] \subseteq f^{-1}[U \cup V]$.

(e) For all $U, V \subseteq Y$, $f^{-1}[U \setminus V] = f^{-1}[U] \setminus f^{-1}[V]$.

The other parts are all also true, and the proofs are similar.