ESP Math 294

# Week 9
## Functions

March 9, 2021

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1. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 2x$. What is $g[\mathbb{Z}]$, that is, the image of $\mathbb{Z}$ under $g$?

2. Let $f : X \to Y$ be any function, and let $U, V \subseteq X$. We proved that it is always true that $f[U \cap V] \subseteq f[U] \cap f[V]$.

   (a) Is it always true that $f[U] \cap f[V] \subseteq f[U \cap V]$? Prove your answer.

   (A useful reminder: in order to prove that it's always true, you should prove the statement only using the fact that $f$ is a function and that $U, V \subseteq X$. But to prove that it's not always true, it's enough to give a single example of a function $f$ and subsets $U, V \subseteq X$ for which the statement doesn't happen. This applies for all the other parts of this problem, as well as other problems on this worksheet.)

   (b) Is it always true that $f[U \cup V] \subseteq f[U] \cup f[V]$? Prove your answer.

   (c) Is it always true that $f[U] \cup f[V] \subseteq f[U \cup V]$? Prove your answer.

3. Let $h : \mathbb{R} \to \mathbb{R}$ be given by $h(x) = |x|$ (remember that $|x|$ is the *absolute value of* $x$, which keeps positive numbers the same but turns negative numbers into positive numbers), and recall that $(a, b)$ is the open interval between $a$ and $b$, that is, it is all real numbers $y$ so that $a < y < b$.

   What is $h^{-1}[(-1, 5)]$? For any two real numbers $a$ and $b$ where $a < b$, what is $h^{-1}[(a, b)]$?

4. Let $f : X \to Y$ be any function. For each of the following statements, determine whether or not the statement is true or false and provide a proof of your answer.

   (a) For all $U, V \subseteq Y$, $f^{-1}[U \cap V] \subseteq f^{-1}[U] \cap f^{-1}[V]$.

   (b) For all $U, V \subseteq Y$,, $f^{-1}[U] \cap f^{-1}[V] \subseteq f^{-1}[U \cap V]$.

   (c) For all $U, V \subseteq Y$, $f^{-1}[U \cup V] \subseteq f^{-1}[U] \cup f^{-1}[V]$.

   (d) For all $U, V \subseteq Y$, $f^{-1}[U] \cup f^{-1}[V] \subseteq f^{-1}[U \cup V]$.

   (e) For all $U, V \subseteq Y$, $f^{-1}[U \setminus V] = f^{-1}[U] \setminus f^{-1}[V]$.

   Compared to the results of problem 1, this tells us that preimages behave better with the set operations than images do.

5. **(Challenge)** Let $f : X \to Y$ be a function. For each of the following statements, determine whether or not the statement is true or and provide a proof of your answer.

   (a) For all $U \subseteq X$, $U \subseteq f^{-1}[f[U]]$.

   (b) For all $U \subseteq X$, $f^{-1}[f[U]] \subseteq U$.

   (c) For all $V \subseteq Y$, $V \subseteq f[f^{-1}[V]]$.

   (d) For all $V \subseteq Y$, $f[f^{-1}[V]] \subseteq V$.