Uniquely C_4-Saturated Graphs*

Joshua Cooper[†] John Lenz[‡] Timothy D. LeSaulnier[‡]
Paul S. Wenger[‡] Douglas B. West[‡§]

January 26, 2010

Abstract

For a fixed graph H, a graph G is *uniquely H-saturated* if G does not contain H, but the addition of any edge from $\overline{G}$ to G completes exactly one copy of H. Using a combination of algebraic methods and counting arguments, we determine all the uniquely C_4-saturated graphs; there are only ten of them.

1 Introduction

For a fixed graph H, a graph G is *H-saturated* if G does not contain H but joining any nonadjacent vertices produces a graph that does contain H. Let P_n, C_n, K_n denote the path, cycle, and complete graph with n vertices, respectively. The study of H-saturated graphs began when Turán [5] determined the n-vertex K_r-saturated graphs with the most edges. In the opposite direction, Erdős, Hajnal, and Moon [1] determined the n-vertex K_r-saturated graphs with the fewest edges. A survey of results and problems about the smallest n-vertex H-saturated graphs appears in [4].

A graph G is *uniquely H-saturated* if G is H-saturated and the addition of any edge joining nonadjacent vertices completes exactly one copy of H. The graphs found in [1] are uniquely K_r-saturated. For example, consider $H = C_3$. Every C_3-saturated graph has diameter at most 2. All trees with diameter 2 are stars and are uniquely C_3-saturated. A uniquely C_3-saturated graph G cannot contain a 3-cycle or a 4-cycle, so such a graph that is not a tree

*Research of all authors partially supported by NSF grant DMS 08-38434 "EMSW21-MCTP: Research Experience for Graduate Students."

[†]Mathematics Dept., Univ. of South Carolina, Columbia, SC; `cooper@math.sc.edu`.

[‡]Mathematics Dept., Univ. of Illinois, Urbana, IL; email addresses `jlenz2@illinois.edu`, `tlesaul2@illinois.edu`, `pwenger2@illinois.edu`, `west@math.uiuc.edu`.

[§]Research supported by NSA grant H98230-09-1-0363.

has girth 5. Every graph with girth 5 and diameter 2 is uniquely C_3-saturated. The graphs with diameter d and girth $2d+1$ are the *Moore graphs*. Hoffman and Singleton [2] proved that besides odd cycles there are only finitely many Moore graphs, all having diameter 2. Thus, except for stars, there are finitely many uniquely C_3-saturated graphs.

Ollmann [3] determined the C_4-saturated n-vertex graphs with the fewest edges, but few of these are uniquely C_4-saturated. An exception is the triangle K_3; whenever $n < |V(H)|$, vacuously K_n is uniquely H-saturated. In this paper we determine all the uniquely C_4-saturated graphs.

Theorem 1. *There are precisely ten uniquely C_4-saturated graphs.*

In the list, the only example with girth 5 is the 5-cycle. The others are small trees or contain triangles; all have at most nine vertices.

The sense in which uniquely C_k-saturated graphs can be viewed as generalizing the Moore graphs of diameter 2 is reflected in our proof. The structure and techniques of the paper are very similar to the eigenvalue approach used to prove both the Hoffman-Singleton result on Moore graphs and the "Friendship Theorem", which states that a graph in which any two distinct vertices have exactly one common neighbor has a vertex adjacent to all others (see Wilf [7]). Structural arguments are used to show that under certain conditions the graphs in question are regular. Counting of walks then yields a polynomial equation involving the adjacency matrix, after which eigenvalue arguments exclude all but a few graphs.

The graphs that result from the Friendship Theorem consist of some number of triangles sharing a single vertex; such graphs are uniquely C_5-saturated. Thus, unlike for C_4, there are infinitely many uniquely C_5-saturated graphs. Wenger [6] has shown that except for small complete graphs, the "friendship graphs" are the only uniquely C_5-saturated graphs.

2 Structural Properties

Our graphs have no loops or multi-edges. A *k-cycle* is a cycle with k vertices, and we define a *k-path* to be a k-vertex path. A path with endpoints x and y is an *x,y-path*. For a vertex v in a graph G, the *neighborhood* $N(v)$ is $\{u \in V(G) \colon uv \in E(G)\}$. The *$k$th neighborhood* $N^k(v)$ is $\{u \in V(G) \colon d(u,v) = k\}$, where the *distance* $d(u,v)$ is the minimum length of a u,v-path. The *diameter* of a graph is the maximum distance between vertices. The *degree* $d(v)$ of a vertex v in a graph G is the number of incident edges.

We begin with basic observations about the structure of uniquely C_4-saturated graphs.

Lemma 2. *The following properties hold for every uniquely C_4-saturated graph G.*

(a) G is connected and has diameter at most 3.

(b) Any two nonadjacent vertices in G are the endpoints of exactly one 4-path.

(c) G contains no 6-cycle and no two triangles sharing a vertex.

Proof. If x and y are nonadjacent vertices in G, then the edge xy completes a 4-cycle. Thus G contains an x, y-path of length 3. Since G is uniquely C_4-saturated, x and y are the endpoints of exactly one 4-path. Opposite vertices on a 6-cycle would be the endpoints of two 4-paths if nonadjacent and would lie on a 4-cycle if adjacent. The same is true for nonadjacent vertices in the union of two triangles sharing one vertex. The union of two triangles sharing two vertices contains a 4-cycle. □

Lemma 3. *If G is uniquely C_4-saturated and $|V(G)| \geq 3$, then G has girth 3 or 5.*

Proof. If G contains a triangle, then G has girth 3, so we may assume that G is triangle-free. Hence there are vertices x and y with $d(x, y) = 2$; let z be their unique common neighbor. By Lemma 2, there is a 4-path joining x and y. If it contains z, then G contains a triangle. Otherwise, x and y lie on a 5-cycle. Since G is C_4-free, it follows that G has girth 5. □

If G has maximum degree at most 1, then G is K_1 or K_2, and these are uniquely C_4-saturated. We may assume henceforth maximum degree at least 2. Lemma 3 then allows us to break the study of uniquely C_4-saturated graphs into two cases: girth 3 and girth 5.

3 Girth 5

Lemma 4. *If G is a uniquely C_4-saturated graph with girth 5, then G is regular.*

Proof. Let u and v be adjacent vertices, with $d(u) \leq d(v)$. Since G is triangle-free, $N(v)$ is an independent set, and hence the 4-paths joining neighbors of v do not contain v. If $d(u) < d(v)$, then by the pigeonhole principle two of the unique 4-paths from u to the other $d(v) - 1$ neighbors of v begin along the same edge uu' incident to u. Each of these two paths continues along an edge to v to form distinct 4-paths from u' to v. Since $N(v)$ is independent, u' is not adjacent to v, so this contradicts Lemma 2.

We conclude that adjacent vertices in G have the same degree. Since G is connected, it follows that G is k-regular. □

We now show that exactly one uniquely C_4-saturated graph has girth 5.

Theorem 5. *The only uniquely C_4-saturated graph with girth 5 is C_5.*

Proof. Let G be a uniquely C_4-saturated n-vertex graph with girth 5. By Lemma 4, G is regular; let k be the vertex degree. Let A be the adjacency matrix of G, let J be the n-by-n matrix with every entry 1, and let $\mathbf{1}$ be the n-vector with each coordinate 1. If x and y are nonadjacent vertices of G, then by Lemma 2 there is one x, y-path of length 3 and no other walk of length 3 joining x and y. If x and y are adjacent, then there are $2k-1$ walks of length 3 joining them. If $x = y$, then no walk of length 3 joins x and y, because G is triangle-free. This yields $A^3 = (J - A - I) + (2k-1)A$, or $J = A^3 - (2k-2)A + I$.

Because J is a polynomial in A, every eigenvector of A is also an eigenvector of J. Since G is k-regular, $\mathbf{1}$ is an eigenvector of A with eigenvalue k. Also $\mathbf{1}$ is an eigenvector of J with eigenvalue n. This yields the following count of the vertices of G:

$$n = k^3 - (2k-2)k + 1 = k^3 - 2k^2 + 2k + 1.$$

We have observed that every eigenvector of A is also an eigenvector of J. Since J has rank 1, we conclude that $Jx = 0x$ when x is an eigenvector of A other than $\mathbf{1}$. If λ is the corresponding eigenvalue of A, then $J = A^3 - (2k-2)A + I$ yields

$$0 = \lambda^3 - (2k-2)\lambda + 1. \tag{1}$$

It follows that A has at most three eigenvalues other than k.

Let q denote the polynomial in (1). Being a cubic polynomial, it factors as

$$q(\lambda) = \lambda^3 - (2k-2)\lambda + 1 = (\lambda - r_1)(\lambda - r_2)(\lambda - r_3). \tag{2}$$

It follows that

$$r_1 + r_2 + r_3 = 0. \tag{3}$$

Suppose first that two of these roots have a common value, r. From (3), the third is $-2r$, and we have

$$\lambda^3 - (2k-2)\lambda + 1 = (\lambda - r)^2(\lambda + 2r) = \lambda^3 - 3r^2\lambda + 2r^3.$$

By equating coefficients, r equals both $(1/2)^{1/3}$ (irrational) and $(2k-2)/3$ (rational). Hence q has three distinct roots.

Suppose next that q has a rational root. The Rational Root Theorem implies that 1 and -1 are the only possible rational roots of q. If -1 is a root, then $k = 1$ and G does not have girth 5. If 1 is a root, then $k = 2$ and $G = C_5$.

Hence we may assume that q has three distinct irrational roots. In this case we will obtain a contradiction. Index the eigenvalues so that the multiplicities a, b, and c of r_1, r_2, and r_3 (respectively) satisfy $a \le b \le c$. Letting p_A be the characteristic polynomial of A,

$$p_A(\lambda) = (\lambda - k)(\lambda - r_1)^a(\lambda - r_2)^b(\lambda - r_3)^c. \tag{4}$$

Combining (2) and (4) yields

$$p_A(\lambda) = (\lambda - k)(\lambda^3 - (2k-2)\lambda + 1)^a(\lambda - r_2)^{b-a}(\lambda - r_3)^{c-a}.$$

Because A has integer entries, $p_A(\lambda) \in \mathbb{Q}[\lambda]$. By applying the division algorithm, $p = rs$ and $p, r \in \mathbb{Q}[\lambda]$ imply $s \in \mathbb{Q}[\lambda]$. Hence $(\lambda - r_2)^{b-a}(\lambda - r_3)^{c-a} \in \mathbb{Q}[\lambda]$. Since $q(\lambda)$ is a monic cubic polynomial in $\mathbb{Q}[\lambda]$ with three irrational roots, it is irreducible and is the minimal polynomial of r_1, r_2, and r_3 over $\mathbb{Q}$. Thus q divides $(\lambda - r_2)^{b-a}(\lambda - r_3)^{c-a}$ if $c > a$. In that case, since r_1 is a root of q, it is also a root of $(\lambda - r_2)^{b-a}(\lambda - r_3)^{c-a}$. We conclude that $c = a$, and all three eigenvalues have the same multiplicity.

The trace of A is 0, so

$$k + ar_1 + ar_2 + ar_3 = k + a(r_1 + r_2 + r_3) = \mathrm{Tr}(A) = 0. \tag{5}$$

Together, (3) and (5) require $k = 0$. Thus q cannot have three distinct irrational roots when G has girth 5. □

4 Girth 3

We now consider uniquely C_4-saturated graphs with a triangle. The next lemma gives a structural decomposition. For a set $S \subseteq V(G)$, let $d(x, S) = \min\{d(x, v) \colon v \in S\}$, let $N(S) = \{v \in V(G) \colon d(v, S) = 1\}$, and let $N^k(S) = \{v \in V(G) \colon d(v, S) = k\}$.

Lemma 6. *Let S be the vertex set of a triangle in a graph G, with $S = \{v_1, v_2, v_3\}$. For $i \in \{1, 2, 3\}$, let $V_i = N(v_i) - S$, and let $V_i' = N^2(v_i) - N(S)$. Let $R = N^3(S)$. If G is uniquely C_4-saturated, then G has the following structure:*

(a) $V_i \cap V_j = \varnothing$ when $i \ne j$;
(b) each vertex in V_i' has exactly one neighbor in V_i;
(c) $V_i' \cap V_j' = \varnothing$ when $i \ne j$;
(d) no edges join V_i' and V_j' when $i \ne j$;
(e) $N(S)$ is independent;
(f) each V_i' induces a matching;
(g) each vertex in R has exactly one neighbor in each V_i'.

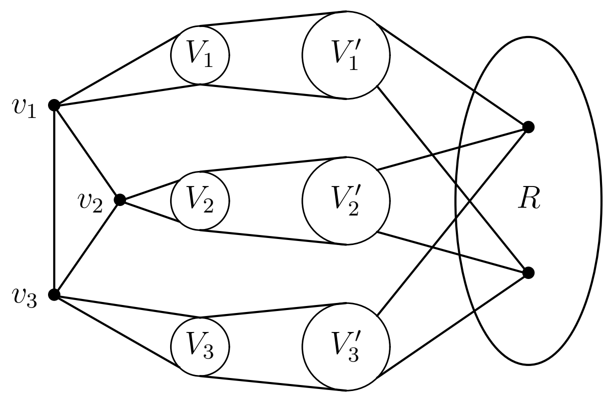

Figure 1: Structure of uniquely C_4-saturated graph with a triangle.

Proof. Since G has diameter 3, we have described all of $V(G)$. Figure 1 makes it easy to see most of the conclusions. The prohibition of 4-cycles and of triangles with common vertices implies (a), (b), and (e). The prohibition of 6-cycles implies (c) and (d).

Given these results, (f) is implied by the existence of a unique 4-path joining v_i to each vertex of V_i'. For (g), each vertex in R is joined by a unique 4-path to each vertex in S; it can only reach v_i quickly enough by moving first to a vertex of V_i', and uniqueness of the 4-path prohibits more than one such neighbor. □

The main part of the argument is analogous to the regularity, walk-counting, and eigenvalue arguments in Lemma 4 and Theorem 5.

Theorem 7. *If G is a C_4-saturated graph with a triangle, then $R = \varnothing$ in the partition of $V(G)$ given in Lemma 6.*

Proof. If $R \neq \varnothing$, then each set V_i and V_i' in the partition is nonempty. We show first that G is regular, then show that each vertex lies in one triangle, and finally count 4-paths to determine the cube of the adjacency matrix and obtain a contradiction using eigenvalues.

Consider V_i' and V_j with $i \neq j$. A vertex x in V_i' reaches each vertex of V_j by a unique 4-path, passing through R and V_j'. By Lemma 6(g), each vertex of R has one neighbor in V_j', so each edge from x to R starts exactly one 4-path to V_j. By Lemma 6, the other neighbors of x are one each in V_i and V_i', so $d(x) = |V_j| + 2$. Since the choice of i and j was arbitrary, we conclude that each vertex of $N^2(S) \cup S$ has degree $a + 2$, where $a = |V_1| = |V_2| = |V_3|$.

For $x \in V_i$ and $y \in V_j$ with $j \neq i$, the unique 4-path joining x to any neighbor of y in V_j' must pass through V_i' and R. By Lemma 6(g), these paths use distinct vertices in R; since

G has no 6-cycle through y, they also use distinct vertices in V_i'. Hence $d(x) \geq d(y)$. By symmetry, all vertices of $N(S)$ have the same degree; let this degree be $b+1$.

Consider $r \in R$. By Lemma 6(g), 4-paths from r to V_i may visit another vertex in R and then reach V_i in exactly one way, or they may go directly to V_i', traverse an edge within V_i', and continue to V_i. The total number of such paths is $[d(r)-3]+1$, and this must equal $|V_i|$. Hence $d(r) = a+2$. Since $|V_i| = a$ and $d(x) = b+1$ for $x \in V_i$, Lemma 6 yields $|V_i'| = ab$.

Consider $x \in V_i'$ and $j \neq i$. Each 4-path from x to V_j' starts with an edge in V_i', ends with an edge in V_j', or uses two vertices in R. Since each vertex in $N^2(S)$ has a neighbors in R, there are a paths of each of the first two types. Since each vertex of R has degree $a+2$, with three neighbors in $N^2(S)$, there are $a(a-1)$ paths of the third type. Since these paths reach distinct vertices of V_j', and every vertex of V_j' is reached, $|V_j'| = a(a+1)$.

Hence $a(a+1) = ab$, and $b = a+1$. Since every vertex of G has degree $a+2$ or $b+1$, we conclude that G is k-regular, where $k = a+2$.

We show next that every vertex of G lies in a triangle. If v lies in no triangle, then $N(v)$ is independent, and having unique 4-paths from $N^2(v)$ to v forces $N^2(v)$ to induce a 1-regular subgraph. Since $|N^2(v)| = k(k-1)$, there are $\binom{k}{2}$ edges induced by $N^2(v)$. Each 4-path with both endpoints in $N(v)$ has internal vertices in $N^2(v)$. Since there are $\binom{k}{2}$ such pairs of endpoints and each edge within $N^2(v)$ extends to exactly one such path, no edge within $N^2(v)$ lies in a triangle with a vertex of $N(v)$. Thus each neighbor of v also lies in no triangle.

We conclude that neighboring vertices both do or both do not lie in triangles. By induction on the distance from S, every vertex lies in a triangle. By Lemma 2, each vertex lies in exactly one triangle.

With A being the adjacency matrix of G, the matrix A^3 again counts walks of length 3. Since each vertex is on one triangle, each diagonal entry is 2. Since G is k-regular, entries for adjacent vertices are $2k-1$, and by unique C_4-saturation the remaining entries equal 1. Hence $A^3 = J + (2k-2)A + I$, and again J is expressible as a polynomial in A:

$$J = A^3 - (2k-2)A - I.$$

Again $\mathbf{1}$ is an eigenvector of A with eigenvalue k and of J with eigenvalue n. All other eigenvalues of A satisfy $p(\lambda) = 0$, where

$$p(\lambda) = \lambda^3 - (2k-2)\lambda - 1.$$

Arguing as in the proof of Theorem 5, $p(\lambda)$ cannot be irreducible over $\mathbb{Q}$. If λ is rational, then $\lambda = \pm 1$, and $k \in \{1, 2\}$. However, $R \neq \varnothing$ requires $k \geq 3$. $\square$

Having shown that $R = \varnothing$, we now consider instances with $N^2(S) \neq \varnothing$.

Lemma 8. *Let G be a uniquely C_4-saturated graph with a triangle having vertex set S. If $N^2(S) \neq \varnothing$, then G is one of the three graphs in Figure 2.*

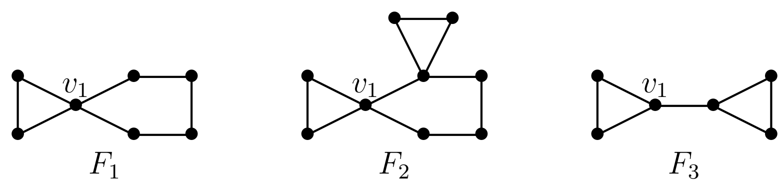

Figure 2: Examples having a vertex at distance 2 from a triangle.

Proof. Let $S = \{v_1, v_2, v_3\}$. In the partition defined in Lemma 6, a 4-path joining V_i' and V_j' must pass through R. Since $R = \varnothing$, we conclude that only one of $\{V_1', V_2', V_3'\}$ is nonempty; by symmetry, let it be V_1'. Since G has diameter 3, we have $V_2 = V_3 = \varnothing$.

By Lemma 6(f), V_1' induces a matching. By Lemma 6(b), every vertex of V_1' thus has degree 2. Consider $w \in V_1$ with neighbors u and v in V_1'. If u and v are not adjacent, then a 4-path joining them must use w and the neighbor in V_1' of one of them. Thus if w has three pairwise nonadjacent neighbors in V_1', then at least two of them have neighbors in V_1' that are also neighbors of w. This yields two triangles containing w, contradicting Lemma 2. We conclude that w cannot have more than three neighbors in V_1'.

If $w \in V_1$ has three neighbors in V_1', then two of them (say x and y) are adjacent. The only 4-paths that can leave x or y for other vertices of V_1' end at the remaining neighbor of w or its mate in V_1'. Hence $G = F_2$.

If $w \in V_1$ has two neighbors in V_1', then they are adjacent, and no 4-paths can join them to other vertices of V_1'. Hence $G = F_3$.

In the remaining case, every vertex of V_1 has at most one neighbor in V_1'. Since any two vertices of V_1 are joined by a 4-path through an edge within V_1', there can only be two vertices in V_1, and $G = F_1$. $\square$

One case remains.

Lemma 9. *If G is a uniquely C_4-saturated graph having a triangle S adjacent to all vertices, then G consists of S and a matching joining S to the remaining (at most three) vertices.*

Proof. We have assumed $N^2(S) = \varnothing$. Since 4-paths joining vertices in V_i must pass through V_i', each V_i has size 0 or 1. Since $V_i \cap V_j = \varnothing$ (Lemma 6(a)), G is as described. $\square$

We can now prove Theorem 1.

Theorem 1. *There are exactly ten uniquely C_4-saturated graphs.*

Proof. Trivially, K_1, K_2, and K_3 are uniquely C_4-saturated. With girth 5, there is only C_5, by Theorem 5. With girth 3, Lemma 8 provides three graphs when some vertex has distance 2 from a triangle, and Lemma 9 provides three when there is no such vertex. □

References

[1] P. Erdős, A. Hajnal, and J. W. Moon, A problem in graph theory. *Amer. Math. Monthly* 71 (1964), 1107–1110.

[2] A. J. Hoffman and R. R. Singleton, On Moore graphs with diameters 2 and 3. *IBM J. Res. Develop.* 4 (1960), 497–504.

[3] L. T. Ollmann, $K_{2,2}$-saturated graphs with a minimal number of edges. *Proc. Third SE Conf. on Combinatorics, Graph Theory, and Computing* (Florida Atlantic Univ., Boca Raton, Fla., 1972), 367–392.

[4] O. Pikhurko, Results and open problems on minimum saturated hypergraphs. *Ars Combin.* 72 (2004), 111–127.

[5] P. Turán, Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok* 48 (1941), 436–452.

[6] P. S. Wenger, Uniquely C_k-saturated graphs. In preparation.

[7] H. S. Wilf, The friendship theorem. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)* (Academic Press, 1971), 307–309.