Solutions to CS / MCS 401 Week #2 Exercises(Fall 2007)

Exercise 3.1-4 (page 50)

$\lim_{n\to\infty} 2^{n+1} / 2^n = \lim_{n\to\infty} 2 = 2$, so $2^{n+1} = O(2^n)$.

$\lim_{n\to\infty} 2^{2n} / 2^n = \lim_{n\to\infty} 2^n = \infty$, so $2^{2n} \neq O(2^n)$.

Exercise 3.2-6 (page 57)

We prove by induction on i that $F_i = (\varphi^i + \hat{\varphi})^i / \sqrt{5}$.

If $i = 0$, $(\varphi^i - \hat{\varphi}^i) / \sqrt{5} = (1 - 1) / \sqrt{5} = 0 = F_0$.
If $i = 1$, $(\varphi^i - \hat{\varphi}^i) / \sqrt{5} = \big((1 + \sqrt{5}\,)/2 - (1 - \sqrt{5}\,)/2 \big) / \sqrt{5} = 1 = F_1$.

Now assume $i \geq 2$, and that the result holds for all k with $k < i$. Recall φ and $\hat{\varphi}$ are the roots of $x^2 - x - 1 = 0$. Thus $\varphi^2 = \varphi + 1$ and $\hat{\varphi}^2 = \hat{\varphi} + 1$. Multiplying by φ^{i-2} gives $\varphi^i = \varphi^{i-1} + \varphi^{i-2}$ and $\hat{\varphi}^i = \hat{\varphi}^{i-1} + \varphi^{i-2}$ for all $i \geq 2$.

$$
\begin{aligned}
F_i = F_{i-1} + F_{i-2} &= (\varphi^{i-1} - \hat{\varphi}^{i-1})/\sqrt{5} + (\varphi^{i-2} - \hat{\varphi}^{i-2})/\sqrt{5} && \text{(inductive hypothesis)} \\
&= (\varphi^{i-1} + \varphi^{i-2})/\sqrt{5} - (\hat{\varphi}^{i-1} + \hat{\varphi}^{i-2})/\sqrt{5} \\
&= \varphi^i/\sqrt{5} - \hat{\varphi}^i/\sqrt{5} \\
&= (\varphi^i - \hat{\varphi}^i)/\sqrt{5}
\end{aligned}
$$

So the result holds for i.

Problem 3-2 (pages 57-58)

A	B	*O*	*o*	Ω	ω	Θ	Notes
$\lg^k(n)$	**n^ε**	yes	yes	no	no	no	Polynomials dominate logarithms.
n^k	**c^n**	yes	yes	no	no	no	Exponentials dominate polynomials.
sqrt(n)	**$n^{\sin(n)}$**	no	no	no	no	no	$n^{\sin(n)}$ oscillates in $(1/n, n)$. There are arbitrarily large values for n for which $n^{\sin(n)} < 1$, and arbitrarily large n for which $n^{\sin(n)} > n^{0.75}$. For any positive constants c_1 and c_2, there exist arbitrarily large values of n with sqrt(n) / $n^{\sin(n)} < c_1$, and arbitrarily large n with sqrt(n) / $n^{\sin(n)} > c_2$.
2^n	**$2^{n/2}$**	no	no	yes	yes	no	$\lim_{n\to\infty} 2^n / 2^{n/2} = \lim_{n\to\infty} 2^{n/2} = \infty$.
$n^{\lg(c)}$	**$c^{\lg(n)}$**	yes	no	yes	no	yes	We showed in class that $n^{\lg(c)} = c^{\lg(n)}$.
lg(n!)	**$\lg(n^n)$**	yes	no	yes	no	yes	$\lg(n^n) = n\lg(n)$, and $\lg(n!) \approx n\lg(n) - 1.44n$ by Stirling's formula.

Problem 3-3 (pages 57-58), part (a)

$2^{2^{n+1}}$

2^{2^n}

$(n+1)!$

$n!$ Note $n! \approx (n/e)^n \text{sqrt}(2\pi n)$ grows more rapidly than α^n, α fixed.

e^n

$n \cdot 2^n$

2^n

$(3/2)^n$

$n^{\lg\lg(n)}$, $(\lg(n))^{\lg(n)}$ Note $(\lg(n))^{\lg(n)} = 2^{\lg\lg(n)\lg(n)} = 2^{\lg(n)\lg\lg(n)} = n^{\lg\lg(n)}$.

$(\lg(n))!$ Note $(\lg(n))! \approx (\lg(n)/e)^{\lg(n)} \text{sqrt}(2\pi\lg(n)) = n^{\lg\lg(n)}/n^{\lg(e)}\,\Theta(\text{sqrt}(\lg(n))$
$= n^{\lg\lg(n)-1.44}\Theta(\text{sqrt}(\lg(n))$.

n^3

n^2, $4^{\lg(n)}$ Note $4^{\lg(n)} = n^{\lg(4)} = n^2$ by the general rule $a^{\log_b(c)} = c^{\log_b(a)}$.

$n\lg(n)$, $\lg(n!)$

n, $2^{\lg(n)}$

$(\text{sqrt}(2))^{\lg(n)}$ $(\text{sqrt}(2))^{\lg(n)} = n^{\lg(\text{sqrt}(2))} = n^{0.5}$

$2^{\text{sqrt}(2\lg(n))}$

$(\lg(n))^2$

$\ln(n)$

$\text{sqrt}(\lg(n))$

$\ln\ln n$

1, $n^{1/\lg(n)}$ Note $n^{1/\lg(n)} = (2^{\lg(n)})^{1/\lg(n)} = 2^1 = 2$.

Problem 3-4 (page 59)

Part (c)

The conjecture

$f(n) = O(g(n))$ implies $\lg(f(n)) = O(\lg(g(n)))$, where $\lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n is true.

$f(n) = O(g(n)) \Rightarrow f(n) \leq c\,g(n))$ For some constant c, this holds for all n sufficiently large. (Increasing c if needed, we may assume $c \geq 1$.)

$\Rightarrow \lg(f(n)) \leq \lg(c\,g(n))$ Since $\lg(x)$ in an increasing function.

$\Rightarrow \lg(f(n)) \leq \lg(c) + \lg(g(n))$

$\Rightarrow \lg(f(n)) \leq \lg(c)\lg(g(n)) + \lg(g(n))$ Since $\lg(g(n)) \geq 1$

$\Rightarrow \lg(f(n)) \leq (\lg(c) + 1)\lg(g(n))$

$\Rightarrow \lg(f(n)) = O(\lg(g(n)))$.

Part (d)

The conjecture

$$f(n) = O(g(n)) \text{ implies } 2^{f(n)} = O\left(2^{g(n)}\right)$$

is false. (It is even false if we assume $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$.)

As a counterexample, we may take $g(n) = n$ and $f(n) = 2n$. Obviously $2n = O(n)$, but $2^{2n} \neq O(2^n)$ since $\lim_{n\to\infty} 2^{2n} / 2^n = \lim_{n\to\infty} 2^n = \infty$.

Exercise C.

Stirling's formula approximates 46! by $(46/e)^{46}$sqrt($2\pi\cdot 46$) $\approx 5.492663 \times 10^{57}$. We obtain a more accurate approximation by multiplying by $1 + 1/(12\cdot 32)$; the result is 5.502613×10^{57}. (The actual value of 46! to seven significant figures is 2.502622×10^{57}.)