Solutions to CS / MCS 401 Exercise Set #2 (Summer 2007)

Problem 3-2 (pages 57-58)

A	B	O	o	Ω	ω	Θ	Notes
$\lg^k(n)$	n^ε	yes	yes	no	no	no	Polynomials dominate logarithms.
n^k	c^n	yes	yes	no	no	no	Exponentials dominate polynomials.
sqrt(n)	$n^{\sin(n)}$	no	no	no	no	no	$n^{\sin(n)}$ oscillates in $(1/n, n)$. There are arbitrarily large values for n for which $n^{\sin(n)} < 1$, and arbitrarily large n for which $n^{\sin(n)} > n^{0.75}$. For any positive constants c_1 and c_2, there exist arbitrarily large values of n with $\text{sqrt}(n) / n^{\sin(n)} < c_1$, and arbitrarily large n with $\text{sqrt}(n) / n^{\sin(n)} > c_2$.
2^n	$2^{n/2}$	no	no	yes	yes	no	$\lim_{n\to\infty} 2^n / 2^{n/2} = \lim_{n\to\infty} 2^{n/2} = \infty$.
$n^{\lg(c)}$	$c^{\lg(n)}$	yes	no	yes	no	yes	We showed in class that $n^{\lg(c)} = c^{\lg(n)}$.
$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes	$\lg(n^n) = n\lg(n)$, and $\lg(n!) \approx n\lg(n) - 1.44n$ by Stirling's formula.

Problem 3-3 (pages 57-58), part (a)

$2^{2^{n+1}}$

2^{2^n}

$(n+1)!$

$n!$ — Note $n! \approx (n/e)^n \,\text{sqrt}(2\pi n)$ grows more rapidly than α^n, α fixed.

e^n

$n \cdot 2^n$

2^n

$(3/2)^n$

$n^{\lg\lg(n)}$, $(\lg(n))^{\lg(n)}$ — Note $(\lg(n))^{\lg(n)} = 2^{\lg\lg(n)\lg(n)} = 2^{\lg(n)\lg\lg(n)} = n^{\lg\lg(n)}$.

$(\lg(n))!$ — Note $(\lg(n))! \approx (\lg(n)/e)^{\lg(n)} \,\text{sqrt}(2\pi\lg(n)) = n^{\lg\lg(n)}/n^{\lg(e)}\,\Theta(\text{sqrt}(\lg(n)))$
$= n^{\lg\lg(n)-1.44}\Theta(\text{sqrt}(\lg(n)))$.

n^3

n^2, $4^{\lg(n)}$ — Note $4^{\lg(n)} = n^{\lg(4)} = n^2$ by the general rule $a^{\log_b(c)} = c^{\log_b(a)}$.

$n\lg(n)$, $\lg(n!)$

n, $2^{\lg(n)}$

$(\text{sqrt}(2))^{\lg(n)}$ — $(\text{sqrt}(2))^{\lg(n)} = n^{\lg(\text{sqrt}(2))} = n^{0.5}$

$2^{\text{sqrt}(2\lg(n))}$

$(\lg(n))^2$

$\ln(n)$

$\text{sqrt}(\lg(n))$

$\ln \ln n$

$1,\quad n^{1/\lg(n)}$ Note $n^{1/\lg(n)} = (2^{\lg(n)})^{1/\lg(n)} = 2^1 = 2.$

Problem 3-4 (page 59)

Part (c)

The conjecture

$f(n) = O(g(n))$ implies $\lg(f(n)) = O(\lg(g(n)))$, where $\lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n is true.

$f(n) = O(g(n)) \Rightarrow f(n) \leq c\,g(n))$ — For some constant c, this holds for all n sufficiently large. (Increasing c if needed, we may assume $c \geq 1$.)

$\Rightarrow \lg(f(n)) \leq \lg(c\,g(n))$ — Since $\lg(x)$ in an increasing function.

$\Rightarrow \lg(f(n)) \leq \lg(c) + \lg(g(n))$

$\Rightarrow \lg(f(n)) \leq \lg(c)\lg(g(n)) + \lg(g(n))$ — Since $\lg(g(n)) \geq 1$

$\Rightarrow \lg(f(n)) \leq (\lg(c) + 1) \lg(g(n))$

$\Rightarrow \lg(f(n)) = O(\lg(g(n)))$.

Part (d)

The conjecture

$f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$

is false. (It is even false if we assume $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$.)

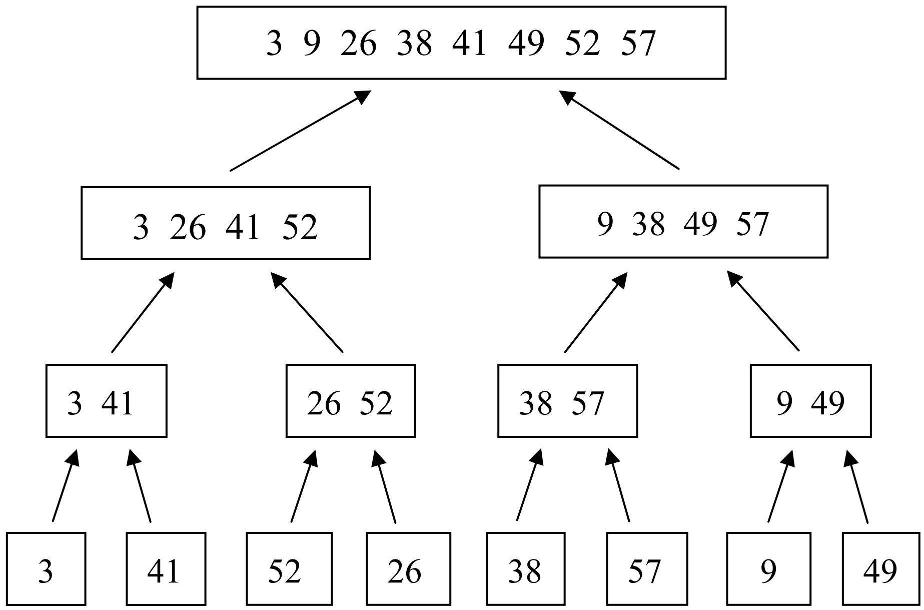
As a counterexample, we may take $g(n) = n$ and $f(n) = 2n$. Obviously $2n = O(n)$, but $2^{2n} \neq O(2^n)$ since $\lim_{n\to\infty} 2^{2n} / 2^n = \lim_{n\to\infty} 2^n = \infty$.

Exercise C.

Stirling's formula approximates 32! by $(32/e)^{32}\text{sqrt}(2\pi\cdot 32) \approx 2.62447\times 10^{35}$. We obtain a more accurate approximation by multiplying by $1 + 1/(12\cdot 32)$; the result is 2.63130×10^{32}. (The actual value of 32! to six decimal places is 2.63131×10^{32}.)

Note: In modifying a previously-given exercise, I changed one but not both occurrences of 40 to 32. In case you computed 40!, the result is $(40/e)^{40}\,\text{sqrt}(2\pi\cdot 40) \approx 8.1422\times 10^{47}$. We obtain a more accurate approximation by multiplying by $1 + 1/(12\cdot 40)$; the result is 8.1591×10^{47}.

Exercise 2.3-1



D. C(0) = 0 = $d \cdot 0$, so the result holds when $n = 0$. Let $n \geq 1$, and assume the result holds for all i with $i < n$.

$C(n) = d + C(k) + C(n-k)$ where $0 \leq k \leq n-1$. Note $n-k-1 \leq n-1$.
By the inductive hypothesis, $C(n) = d + dk + d(n-k-1) = d(1 + k + (n-k-1)) = dn$, so the result also holds for n. By induction it holds for all nonnegative integers.

E. In each part, we assume $n = 2^k$, so $k = \lg(n)$.

a)

$$
\begin{aligned}
C(n) &= C(n/2) + 2n + 3 \\
&= \big(C(n/2^2) + 2(n/2) + 3\big) + 2n + 3 \\
&= C(n/2^2) + 2(n/2 + n) + 2\cdot 3 \\
&= \big(C(n/2^3) + 2(n/2^2) + 3\big) + 2(n/2 + n) + 2\cdot 3 \\
&= C(n/2^3) + 2(n/2^2 + n/2 + n) + 3\cdot 3 \\
&\ \ \vdots \\
&= C(n/2^k) + 2(n/2^{k-1} + \ldots + n/2^2 + n/2 + n) + k\cdot 3 \\
&= C(1) + 2n(1/2^{k-1} + \ldots + 1/2^2 + 1/2 + 1) + k\cdot 3 \\
&= 1 + 2n(2 - 1/2^{k-1}) + 3\lg(n) \\
&= 1 + 4n - 4 + 3\lg(n) \qquad \text{(since } n/2^{k-1} = 2^k/2^{k-1} = 2\text{)} \\
&= \mathbf{4n - 3 + 3\lg(n)}
\end{aligned}
$$

b) $C(n) = 2C(n/2) + n\lg(n)$

$$= 2\big(2C(n/2^2) + n/2\cdot\lg(n/2)\big) + n\lg(n)$$
$$= 2^2C(n/2^2) + n(\lg(n) - 1) + n\lg(n)$$
$$= 2^2C(n/2^2) + 2n\lg(n) - n$$
$$= 2^2\big(2C(n/2^3) + n/2^2\cdot\lg(n/2^2)\big) + 2n\lg(n) - n$$
$$= 2^3C(n/2^3) + n(\lg(n) - 2) + 2n\lg(n) - n$$
$$= 2^3C(n/2^3) + 3n\lg(n) - n(1+2)$$
$$\vdots$$
$$= 2^kC(n/2^k) + kn\lg(n) - n(1+2+\ldots+k-1)$$
$$= nC(1) + n(\lg(n))^2 - nk(k-1)/2$$
$$= n(\lg(n))^2 - n\lg(n)(\lg(n)-1)/2$$
$$= \boldsymbol{n(\lg(n))^2/2 + n(\lg(n))/2}$$

c) *To be filled in.*

Problem 4-4, parts (a), (c), (e), (h)

a) $T(n) = 3T(n/2) + n\lg(n)$.
In the Master Theorem, $a = 3$, $b = 2$, $E = \lg(3) \approx 1.59$, and $f(n) = n\lg(n)$. $f(n) = O(n^{E-\varepsilon})$, where we could take $\varepsilon = 0.1$. The Master Theorem (case 1) tells us $T(n) = \Theta(n^{\lg(3)}) \approx \boldsymbol{\Theta(n^{1.59})}$**.**

c) $T(n) = 4T(n/2) + n^2 sqrt(n) = 4T(n/2) + n^{5/2}$.
In the Master Theorem, $a = 4$, $b = 2$, $E = \lg(4) = 2$, $n^E = n^2$, and $f(n) = n^{5/2}$. $f(n) = \Omega(n^{E+\varepsilon})$, where we could take $\varepsilon = 0.1$. $af(n/b) = 4(n/2)^{5/2} = 2^{-1/2}n$, so $af(n/b) \le cf(n)$ where $c = 2^{-1/2} < 1$. The Master Theorem (case 3) tells us $T(n) = \Theta(f(n)) = \boldsymbol{\Theta(n^{5/2})}$**.**

e) $T(n) = 2T(n/2) + n/\lg(n)$.

In the notation of the Master Theorem, $a = 2$, $b = 2$, $E = \lg(2) = 1$, $f(n) = n/\lg(n)$, and $n^E = n$. The Master Theorem does not apply, as f(n) grows too rapidly for case (1) and not rapidly enough for case (2). But the extension to the Master Theorem in the handout tells us that T(n) = $\boldsymbol{\Theta(n\,\lg\lg(n))}$**.**

h) $T(n) = T(n-1) + \lg(n)$

$$T(n) = T(n-1) + \lg(n)$$
$$= T(n-2) + \lg(n-1) + \lg(n)$$
$$= T(n-3) + \lg(n-2) + \lg(n-1) + \lg(n)$$
$$\vdots$$
$$= T(1) + \lg(2) + \lg(n-2) + \lg(n-1) + \lg(n)$$
$$= T(1) + \lg(n!)$$
$$= n\lg(n) - 1.44n + O(\lg(n))$$
$$= \Theta(n\lg(n)$$