All-Pairs Shortest Paths

Problem: G is a weighted graph or digraph with n vertices, which for simplicity we label simply $1,2,\ldots,n$.

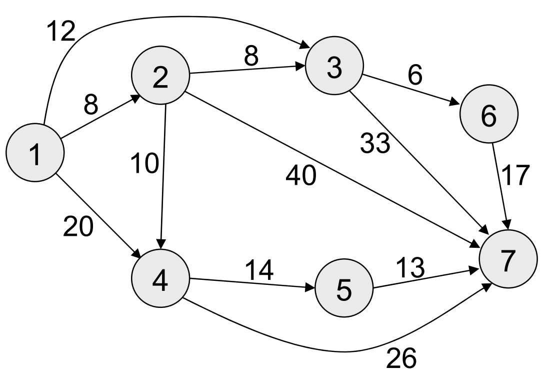
We are given the adjacency matrix $W = (w_{ij})$, w_{ij} = weight of edge from i to j (∞ if no such edge, 0 if $i = j$). All weights are positive.

Find the distance matrix $D = (d_{ij})$, d_{ij} = distance from i to j.

Idea: For $k = 0, 1, \ldots, n$, let

$short_k(i,j)$ = shortest path from i to j all of whose intermediate vertices lie in the set $\{1,2,\ldots,k\}$.

$d_{ij}^{\,k}$ = *length of* $short_k(i,j)$.



$d_{17}^{\,0} = \infty$

$d_{17}^{\,1} = \infty$

$d_{17}^{\,2} = 48$

$d_{17}^{\,3} = 45$

$d_{17}^{\,4} = 44$

$d_{17}^{\,5} = 44$

$d_{17}^{\,6} = 35$

$d_{17}^{\,7} = 35$

$k = 0$: $short_0(i,j)$ = edge from i to j.

$d_{ij}^{\,0} = w_{ij}$.

$k = n$: $short_n(i,j)$ = shortest path from i to j.

$d_{ij}^{\,n} = d_{ij}$.

Initially, we know all the $d_{ij}^{\,0}$.

Our goal is to find all the $d_{ij}^{\,n}$.

How can we find all of the $d_{ij}^{\,k}$, assuming we already know the $d_{ij}^{\,k-1}$?

Case 1: k is not an intermediate vertex on $short_k(i,j)$.

$short_k(i,j) = short_{k-1}(i,j)$

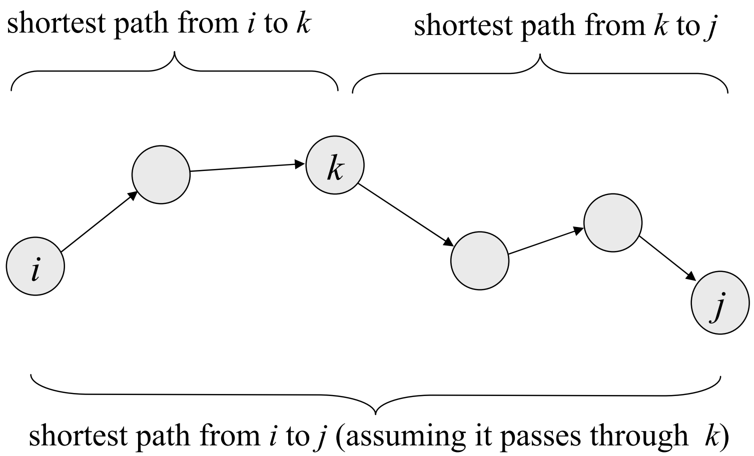
$d_{ij}^{\,k} = d_{ij}^{\,k-1}$.

Vertex k doesn't help. (Always the case if k = i or k = j.)

Case 2: k is an intermediate vertex on $short_k(i,j)$.

$short_k(i,j) = short_{k-1}(i,k) + short_{k-1}(k,j)$.

$d_{ij}^{\,k} = d_{ik}^{\,k-1} + d_{kj}^{\,k-1}$.



Which case applies, case 1 or case 2?

Answer: Whichever minimizes d_{ij}^{k}.

$$d_{ij}^{k} = min(\, d_{ij}^{k-1},\ d_{ik}^{k-1} + d_{kj}^{k-1})$$

If we let

$$p_{ij}^{k} = \begin{cases} \textit{true} & \text{if } d_{ik}^{k-1} + d_{kj}^{k-1} \text{ produces the minimum above,} \\ \textit{false} & \text{otherwise,} \end{cases}$$

then

p_{ij}^{k} is true if and only if k is an intermediate point of $short_k(i,j)$.

Rather than compute all the p_{ij}^{k}, the algorithm below computes

p_{ij} = largest k for which p_{ij}^{k} is true, or 0 if p_{ij}^{k} is false for all k.

Note

p_{ij} =the highest-numbered intermediate point on the shortest path from i to j, or 0 if there are no intermediate points.

We can compute all the d_{ij}^{k} and p_{ij} in $\Theta(n^3)$ time by:

```
for ( i = 1,2,...,n )
    for ( j = 1,2,...,n )
        d_ij^0 = w_ij;
        p_ij = 0;
for ( k = 1,2,...,n )
    for ( i = 1,2,...,n )
        for ( j = 1,2,...,n )
            if ( d_ik^(k-1) + d_kj^(k-1) < d_ij^(k-1) )
                d_ij^k = d_ik^(k-1) + d_kj^(k-1);
                p_ij = k;
            else
                d_ij^k = d_ij^(k-1);
```

The matrix $D = (d_{ij}^{n})$ is the distance matrix, and the matrix $P = (p_{ij})$ has the information needed to find the shortest path between any pair of points.

Using the matrix P, we may print the shortest path from i to j:

```
print(i);
print_intermediate_points( i, j);
print(j);

void print_intermediate_points( int i, int j)
    k = p_ij ;
    if ( k > 0 )
        print_intermediate_points( i, k);
        print( k);
        print_intermediate_points( k, j);
    return;
```

Our algorithm for computing D and P uses $\Theta(n^3)$ space. We can reduce space (but not time) to $\Theta(n^2)$ by updating the $d_{ij}^{\,k}$ in place.

Consider a single pass through the outer loop (fixed k).

d_{ij} doesn't change if i=k or j=k, so there is no problem of using the new value of d_{ik} or d_{kj} when we need the old.

```
for ( i = 1,2,...,n )
    for ( j = 1,2,...,n )
        d_ij = w_ij;
        p_ij = 0;
for ( k = 1,2,...,n )
    for ( i = 1,2,...,n)
        for ( j = 1,2,...,n )
            if ( d_ik + d_kj < d_ij )
                d_ij = d_ik + d_kj;
                p_ij = k;
```