Rate of Growth:
Exponentials, Polynomials, and Logorithms

Theorem 1. [*exponentials dominate polynomials*]
Let a be any fixed real number with $a > 1$, and let n be any fixed real number. Then $\lim_{x\to\infty} \frac{a^x}{x^n} = \infty$.

Proof. We may assume n is an integer, since if the theorem holds for $\lceil n \rceil$, it will hold for n. We use induction on n.

The assertion obviously holds if $n \le 0$, since as $x \to \infty$, $a^x \to \infty$ while x^n is constant ($n = 0$) or $x^n \to 0$ ($n < 0$).

If the theorem fails for some n, choose n minimal such that it fails. By the remark above, $n \ge 1$, and $\lim_{x\to\infty} x^n = \infty$. Since the theorem holds for $n-1$,

$\lim_{x\to\infty} \frac{a^x}{x^{n-1}} = \infty$.

Let $f(x) = a^x$ and $g(x) = x^n$. L'Hopital's rule tells us that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{\ln(a)\, a^x}{nx^{n-1}}$$
$$= (\ln(a)/n) \lim_{x\to\infty} \frac{a^x}{x^{n-1}} = (\ln(a)/n)\,\infty \;= \infty$$

Theorem 2. [*polynomials dominate logarithms*]
Let n be any fixed *positive* real number, and let k be any fixed real number. Then $\lim_{x\to\infty} \frac{x^n}{(\log_c(x))^k} = \infty$.

Proof. Since the ratio of $(\log_c(x))^k$ to $(\ln(x))^k$ is a nonzero constant, namely $(\log_c(e))^k$, we may restrict to the natural logarithm ($c = e$). By the same reasoning as in Theorem 1, we may assume k is an integer. We use induction on k.

Again as in Theorem 1, the theorem is obviously true if $k \le 0$.

If the theorem fails for some k, choose k minimal such that it fails. By the remark above, $k \ge 1$, and $\lim_{x\to\infty} (\ln(x))^k = \infty$. Since the theorem holds for $k-1$,

$\lim_{x\to\infty} \frac{x^n}{\ln(x)^{k-1}} = \infty$.

Let $f(x) = x^n$ and $g(x) = (\ln(x))^k$. L'Hopital's rule tells us that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{nx^{n-1}}{k\ln(x)^{k-1}(1/x)}$$
$$= (n/k) \lim_{x\to\infty} \frac{x^n}{\ln(x)^{k-1}} = (n/k)\,\infty \;= \infty.$$

The following theorem incorporates both Theorems 1 and 2.

Theorem 3. Let a and b be fixed positive real numbers, and let c be a fixed real number with $c > 1$. Then

$$\lim_{x\to\infty} \frac{a^x x^m \log_c(x)^j}{b^x x^n \log_c(x)^k} = \infty$$

if

i) $a > b$, or
ii) $a = b$ and $m > n$, or
iii) $a = b$, $m = n$, and $j > k$.

The limit is 1 if $a = b$, $m = n$, and $j = k$. In all other cases, the limit is 0.

Notes:

1) Theorem 1 tells us, for example, that $1.001^x > x^{100}$ for all x sufficiently large. Moreover, $\lim_{x\to\infty} 1.001^x / x^{100} = \infty$.

   In fact, $1.001^x > x^{100}$ for $x > 1.42 \times 10^6$, approximately. If $x = 2 \times 10^6$, $1.001^x / x^{100} > 10^{238}$.

2) Even an exponential function such as $f(x) = a^{\sqrt{x}}$ will dominate x^n for any fixed n. The same holds if we replace $\sqrt{x}$ by $\sqrt[k]{x}$ for any fixed k.

   On the other hand, $a^{\log_c(x)}$ does not dominate x^n for any fixed n. In fact, $a^{\log_c(x)} = x^k$, where $k = \log_c(a)$.