An Extension to The Master Theorem

In the Master Theorem, as given in the textbook and previous handout, there is a gap between cases (1) and (2), and a gap between cases (2) and (3).

For example, if $a = b = 2$ and $f(n) = n/\lg(n)$ or $f(n) = n\lg(n)$, none of the cases apply. The extension below partially fills these gaps.

THEOREM (*Extension of Master Theorem*) If a, b, $E \stackrel{\text{def}}{=} \log_b(a)$, and $f(n)$ are as in the Master Theorem, the recurrence

$$T(n) = a\,T(n/b) + f(n),\ \ T(1) = d,$$

has solution as follows:

1') If $f(n) = O(n^E(\log_b n)^\alpha)$ with $\alpha < -1$, then $T(n) = \Theta(n^E)$.

2') If $f(n) = \Theta(n^E(\log_b n)^{-1})$, then $T(n) = \Theta(n^E \log_b\log_b(n))$.

3') If $f(n) = \Theta(n^E(\log_b n)^\alpha)$ with $\alpha > -1$, then $T(n) = \Theta(n^E(\log_b n)^{\alpha+1})$.

4') [*same as in Master Theorem*] If $f(n) = \Omega(n^{E+\varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(f(n))$, provided there is a constant c with $c < 1$ for which $a\,f(n/b) \le c\,f(n)$ for all n sufficiently large.

Note: (1') above includes case (1) of the Master Theorem.

(3') above with $\alpha = 0$ is case (2) in the Master Theorem.

We make use of the fact below, which follows from the close connection between sums and integrals.

LEMMA. $\sum_{i=1}^{\infty} i^\alpha$ converges if $\alpha < -1$ and diverges otherwise. $\sum_{i=1}^{n} i^\alpha \approx \ln(n)+\gamma$ if $\alpha = -1$, and $\sum_{i=1}^{n} i^\alpha \approx n^{\alpha+1}/(\alpha+1)$ if $\alpha > -1$.

Proof of the extended Master Theorem when *n* is a power of b.

Case (4) is exactly as in the Master Theorem, so we consider only (1), (2), and (3). In case 1, $f(n) \le \Theta(n^E(\log_b n)^\alpha)$. In cases (2) and (3), $f(n) = \Theta(n^E(\log_b n)^\alpha)$ for some α.

Let $n = b^k$, so $k = \log_b(n)$. From the previous handout, we know that

$$T(n) = f(n) + af(n/b) + a^2 f(n/b^2) + \dots + a^{k-1}f(n/b^{k-1}) + a^k d.$$

Putting $f(n) \approx cn^E(\log_b n)^\alpha$ for some constant c, we get

$$\begin{aligned} T(n) \approx\ & cn^E(\log_b n)^\alpha \\ &+ ac(n/b)^E(\log_b(n/b))^\alpha \\ &+ a^2c(n/b^2)^E(\log_b(n/b^2))^\alpha \\ &+ \dots \\ &+ a^{k-1}c(n/b^{k-1})^E(\log_b(n/b^{k-1}))^\alpha \\ &+ a^k d \end{aligned}$$

(In case 1, this is just an upper bound for $T(n)$.)

Note $a^k = n^E$. Also $n = b^k$, so $\log_b(n/b^i) = \log_b(b^{k-i}) = k-i$. Finally, note $a = b^E$, so in $a^i c(n/b^i)^E$ in the formula above, $a^i \,/\, b^{iE} = 1$.

With these simplifications, our formula becomes

$$T(n) \approx cn^E k^\alpha + cn^E(k-1)^\alpha + cn^E(k-2)^\alpha + \ldots + cn^E 1^\alpha + dn^E$$
$$= cn^E \textstyle\sum_{i=1}^{k} i^\alpha + dn^E$$

If $\alpha < -1$, then $1 \leq \sum_{i=1}^{k} i^\alpha < c'$, where $c' = \sum_{i=1}^{\infty} i^\alpha$ = some constant. So at worst $T(n) \approx (cc'+d)\,n^E = \Theta(n^E)$. But in the handout on the Master Theorem we remarked that $T(n)$ can never be less than $\Theta(n^E)$, since the bottom level alone requires this much time.

If $\alpha = -1$, then $\sum_{i=1}^{k} i^\alpha \approx \ln(k) = \ln \log_b(n) = q \log_b \log_b(n)$ for some constant q, so

$$T(n) \approx cq\,n^E \log_b \log_b(n) + dn^E = \Theta\big(n^E \log_b \log_b(n)\big).$$

If $\alpha > -1$, then $\alpha+1 > 0$, and $\sum_{i=1}^{k} i^\alpha \approx k^{\alpha+1}/(\alpha+1) = \log_b(n)^{\alpha+1}/(\alpha+1)$, so

$$T(n) \approx c\,n^E \log_b(n)^{\alpha+1}/(\alpha+1) + dn^E = \Theta\big(n^E (\log_b(n))^{\alpha+1}\big).$$