Nearly Complete Binary Trees and Heaps

DEFINITIONS:

i) The ***depth*** of a node p in a binary tree is the length (number of edges) of the path from the root to p.

ii) The ***height*** (or ***depth***) of a binary tree is the maximum depth of any node, or -1 if the tree is empty.

Any binary tree can have at most 2^d nodes at depth d. (Easy proof by induction)

DEFINITION: A ***complete binary tree*** of height h is a binary tree which contains exactly 2^d nodes at depth d, $0 \le d \le h$.

- In this tree, every node at depth less than h has two children. The nodes at depth h are the leaves.
- The relationship between n (the number of nodes) and h (the height) is given by

$$n = 1 + 2 + 2^2 + ... + 2^{h-1} + 2^h = 2^{h+1} - 1$$

and

$$h = \lg(n+1) - 1.$$

- Complete binary trees are perfectly balanced and have the maximum possible number of nodes, given their height
- However, they exist only when n is one less than a power of 2.

DEFINITION: A ***nearly complete binary tree*** of height h is a binary tree of height h in which

a) There are 2^d nodes at depth d for $d = 1, 2, ..., h-1$,

b) The nodes at depth h are as far left as possible.

- Condition (b) can be stated more rigorously, like this:

  If a node p at depth $h-1$ has a left child, then every node at depth $h-1$ to the left of p has 2 children. If a node at depth $h-1$ has a right child, then it also has a left child.

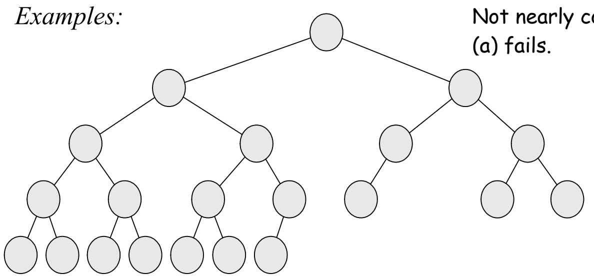
- The relationship between the height and number of nodes in a nearly complete binary tree is given by

$$2^h \le n \le 2^{h+1} - 1, \quad \text{or} \quad h = \lfloor \lg(n) \rfloor.$$

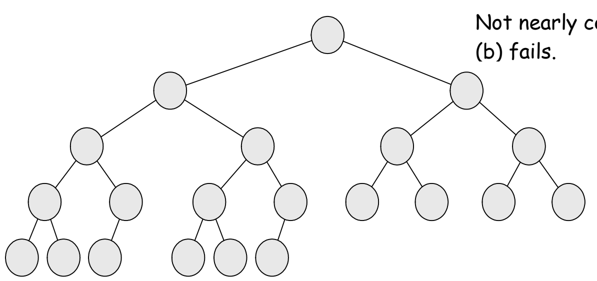
(This depends only on condition (a) in the definition.)

Examples:

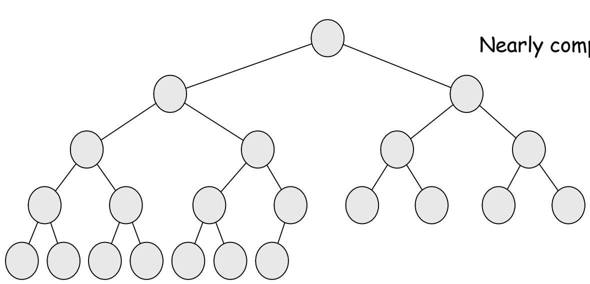
Not nearly complete: (a) fails.



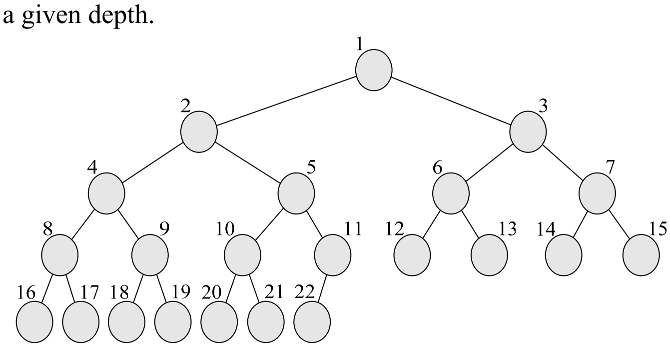
Not nearly complete: (b) fails.



Nearly complete.



Say we label the nodes of a nearly complete binary tree by 1, 2, 3, ..., n in order of increasing depth, and left-to-right at a given depth.



Then, equating each node with its label,

i) $left(k) = 2k$, if $2k \le n$,

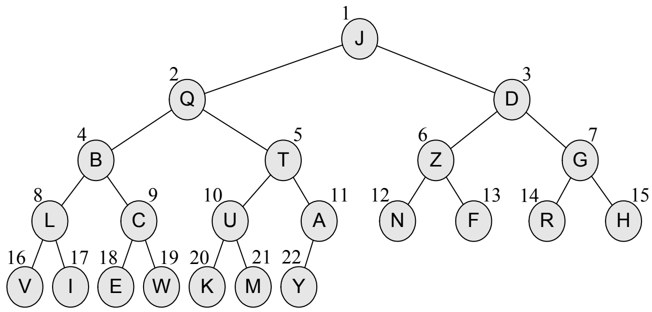
ii) $right(k) = 2k+1$, if $2k+1 \le n$,

iii) $parent(k) = \lfloor k/2 \rfloor$ if $k > 1$.

iv) k has one or more children if $2k \le n$. It has two children if any only if $2k+1 \le n$.

v) k is the left child of its parent if and only if k is even.

Suppose each node in the tree contains an element from some set. Denote the element in node p as *element*(p).



We don't really need the tree structure (nodes with pointers to the two children, and possibly the parent).

We can represent the tree implicitly by an array.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
a	J	Q	D	B	T	Z	G	L	C	U	A	N	F	R	H	V	I	E	W	K	M	Y

The array contains all the information in the tree.

- In the tree, if *p* is the node containing T (node 5), then *parent*(*p*) contains Q, *left*(p) contains U, and *right*(p) contains A. (We examine the link fields in the node.)
- In the array representation, we compute $\lfloor 5/2 \rfloor = 2$, $2\cdot 5 = 10$, and $2\cdot 5+1 == 11$, and we find *parent*(*a*[5]] = *a*[2] = Q, *left*(*a*[5]] = *a*[10] = U, and *right*(*a*[5]) = *a*[11] = A.

It is useful to think in terms of the tree, but all computation is actually performed with the array.

DEFINITION: A ***max-heap*** (or simply a ***heap***) is a nearly complete binary tree in which each node contains an element from a set *S* with a strict weak ordering, such that:

> For each node *p* except the root, *element*(*parent*(*p*)) $\gtrsim$ *element*(*p*). } Heap condition at node p

A ***min-heap*** is defined similarly except the heap condition is *element*(*parent*(*p*)) $\lesssim$ *element*(*p*).

Example of max-heap: