Summation by Parts

An important technique of calculus is integration by parts:

$$\int_a^b u(x)\,v'(x)\,dx \;=\; u(b)\,v(b) - u(a)\,v(a) - \int_a^b u'(x)\,v(x)\,dx$$

This is useful, obviously, when $u'(x)\,v(x)$ is easier to integrate than $u(x)\,v'(x)$, e.g., if $u(x) = x$ and $v(x) = e^x$.

An analogous technique, called summation by parts, works for sums. One version of the summation by parts formula is:

$$\boxed{\sum_{i=a+1}^{b} u_i\,(v_i - v_{i-1}) \;=\; u_b v_b - u_a v_a - \sum_{i=a+1}^{b} (u_i - u_{i-1})\,v_{i-1}}$$

Example: Find $\sum_{i=1}^{n} i2^i$.

Set $a = 0,\ b = n,\ u_i = i,$ and $v_i = 2^{i+1}$.

Then $u_i - u_{i-1} = 1,\ v_i - v_{i-1} = 2^{i+1} - 2^i = 2^i$, and

$$\begin{aligned}
\sum_{i=1}^{n} i2^i \;&=\; \sum_{i=a+1}^{b} u_i\,(v_i - v_{i-1}) \\
&=\; u_b v_b - u_a v_a - \sum_{i=a+1}^{b} (u_i - u_{i-1})\,v_{i-1} \\
&=\; n2^{n+1} - 0\cdot 2^1 - \sum_{i=1}^{n} 1\cdot 2^i \\
&=\; n2^{n+1} - (2^{n+1} - 2) \\
&=\; \mathbf{(n-1)\,2^{n+1} + 2}
\end{aligned}$$