Solutions to CS / MCS 401 Exercise Set #2 (Spring 2008)

Exercise 3.1-4 (page 50)

$\lim_{n\to\infty} 2^{n+1} / 2^n = \lim_{n\to\infty} 2 = 2$, so $2^{n+1} = O(2^n)$.

$\lim_{n\to\infty} 2^{2n} / 2^n = \lim_{n\to\infty} 2^n = \infty$, so $2^{2n} \neq O(2^n)$.

Exercise 3.2-6 (page 57)

We prove by induction on i that $F_i = (\varphi^i + \hat{\varphi})^i / \sqrt{5}$.

If $i = 0$, $(\varphi^i - \hat{\varphi}^i)/\sqrt{5} = (1 - 1)/\sqrt{5} = 0 = F_0$.
If $i = 1$, $(\varphi^i - \hat{\varphi}^i)/\sqrt{5} = \big((1 + \sqrt{5})/2 - (1 - \sqrt{5})/2 \big) / \sqrt{5} = 1 = F_1$.

Now assume $i \geq 2$, and that the result holds for all k with $k < i$. Recall φ and $\hat{\varphi}$ are the roots of $x^2 - x - 1 = 0$. Thus $\varphi^2 = \varphi + 1$ and $\hat{\varphi}^2 = \hat{\varphi} + 1$. Multiplying by φ^{i-2} gives $\varphi^i = \varphi^{i-1} + \varphi^{i-2}$ and $\hat{\varphi}^i = \hat{\varphi}^{i-1} + \varphi^{i-2}$ for all $i \geq 2$.

$$
\begin{aligned}
F_i = F_{i-1} + F_{i-2} &= (\varphi^{i-1} - \hat{\varphi}^{i-1})/\sqrt{5} + (\varphi^{i-2} - \hat{\varphi}^{i-2})/\sqrt{5} \qquad \text{(inductive hypothesis)} \\
&= (\varphi^{i-1} + \varphi^{i-2})/\sqrt{5} - (\hat{\varphi}^{i-1} + \hat{\varphi}^{i-2})/\sqrt{5} \\
&= \varphi^i/\sqrt{5} - \hat{\varphi}^i/\sqrt{5} \\
&= (\varphi^i - \hat{\varphi}^i)/\sqrt{5}
\end{aligned}
$$

So the result holds for i.

Exercise 3-2 (pages 57-58)

A	B	O	o	Ω	ω	Θ	Notes
$\lg^k(n)$	n^ε	yes	yes	no	no	no	Polynomials dominate logarithms.
n^k	c^n	yes	yes	no	no	no	Exponentials dominate polynomials.
sqrt(n)	$n^{\sin(n)}$	no	no	no	no	no	$n^{\sin(n)}$ oscillates in $(1/n, n)$. There are arbitrarily large values for n for which $n^{\sin(n)} < 1$, and arbitrarily large n for which $n^{\sin(n)} > n^{0.75}$. For any positive constants c_1 and c_2, there exist arbitrarily large values of n with sqrt(n) / $n^{\sin(n)} < c_1$, and arbitrarily large n with sqrt(n) / $n^{\sin(n)} > c_2$.
2^n	$2^{n/2}$	no	no	yes	yes	no	$\lim_{n\to\infty} 2^n / 2^{n/2} = \lim_{n\to\infty} 2^{n/2} = \infty$.
$n^{\lg(c)}$	$c^{\lg(n)}$	yes	no	yes	no	yes	We showed in class that $n^{\lg(c)} = c^{\lg(n)}$.
$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes	$\lg(n^n) = n\lg(n)$, and $\lg(n!) \approx n\lg(n) - 1.44n$ by Stirling's formula.

Exercise 3-3 (pages 57-58), part (a)

$2^{2^{n+1}}$

2^{2^n}

$(n+1)!$

$n!$ — Note $n! \approx (n/e)^n \operatorname{sqrt}(2\pi n)$ grows more rapidly than α^n, α fixed.

e^n

$n \cdot 2^n$

2^n

$(3/2)^n$

$n^{\lg\lg(n)}$, $(\lg(n))^{\lg(n)}$ — Note $(\lg(n))^{\lg(n)} = 2^{\lg\lg(n)\lg(n)} = 2^{\lg(n)\lg\lg(n)} = n^{\lg\lg(n)}$.

$(\lg(n))!$ — Note $(\lg(n))! \approx (\lg(n)/e)^{\lg(n)} \operatorname{sqrt}(2\pi\lg(n)) = n^{\lg\lg(n)}/n^{\lg(e)}\,\Theta(\operatorname{sqrt}(\lg(n)))$
$= n^{\lg\lg(n)-1.44}\Theta(\operatorname{sqrt}(\lg(n)))$.

n^3

n^2, $4^{\lg(n)}$ — Note $4^{\lg(n)} = n^{\lg(4)} = n^2$ by the general rule $a^{\log_b(c)} = c^{\log_b(a)}$.

$n\lg(n)$, $\lg(n!)$ — Note $\lg(n!) \approx n\lg(n) - 1.44n$

n, $2^{\lg(n)}$

$(\operatorname{sqrt}(2))^{\lg(n)}$ — Note $(\operatorname{sqrt}(2))^{\lg(n)} = n^{\lg(\operatorname{sqrt}(2))} = n^{0.5}$

$2^{\operatorname{sqrt}(2\lg(n))}$

$(\lg(n))^2$

$\ln(n)$

$\operatorname{sqrt}(\lg(n))$

$\ln\ln n$

1, $n^{1/\lg(n)}$ — Note $n^{1/\lg(n)} = (2^{\lg(n)})^{1/\lg(n)} = 2^1 = 2$.

Exercise C1

a) Recall that $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$. Note $\lim_{n\to\infty} g(n) = \infty$ implies $\ln(g(n)) \geq 1$ for all n sufficently large. Note also $f(n) = O(g(n))$ means that there exists a constant c such that $f(n) \leq c\,g(n))$ for all n sufficiently large. Increasing c if necessary, we may assume $c \geq 1$ and hence $\ln(c) > 0$. Thus for all n sufficiently large,

$$f(n) \leq c\,g(n)) \Rightarrow \ln(f(n)) \leq \ln(c\,g(n)) \quad \text{Since } \ln(x) \text{ in an increasing function.}$$
$$\Rightarrow \ln(f(n)) \leq \ln(c) + \ln(g(n))$$
$$\Rightarrow \ln(f(n)) \leq \ln(c)\ln(g(n)) + \ln(g(n)) \quad \text{Since } \ln(c) > 0 \text{ and } \ln(g(n)) \geq 1.$$
$$\Rightarrow \ln(f(n)) \leq (\ln(c) + 1)\ln(g(n))$$

Since $\ln(c)+1$ is constant, $\ln(f(n)) = O(\ln(g(n)))$.

b) Let $f(n) = n^2$ and $g(n) = n$. Then $\ln(f(n)) = 2\ln(n)$ and $\ln(g(n)) = \ln(n)$, so

$$\ln(f(n)) = 2\ln(g(n)) = O(\ln(g(n))$$

but obviously $f(n) \neq O(g(n))$.

Exercise C2.

Stirling's formula approximates 51! by

$$(51/e)^{51}\text{sqrt}(2\pi\cdot 51) \approx \mathbf{1.5485863 \times 10^{66}}.$$

We obtain a more A more accurate approximation is obtained by multiplying the above by $1 + 1/12n$, giving

$$(51/e)^{51}\text{sqrt}(2\pi\cdot 51)(1+1/(12\cdot 51)) \approx \mathbf{1.5511167 \times 10^{66}}.$$

Note: The actual value of 51! , to 8 significant figures, is 1.5511187×10^{66}.