# Solutions to CS / MCS 401 Week #5 Exercises (Spring 2008)

**Problem 4-4, parts (a), (c), (e), (h)**

**a)** $T(n) = 3T(n/2) + n\lg(n)$.
In the Master Theorem, $a = 3$, $b = 2$, $E = \lg(3) \approx 1.59$, and $f(n) = n\lg(n)$. $f(n) = O(n^{E-\varepsilon})$, where we could take $\varepsilon = 0.1$. The Master Theorem (case 1) tells us $T(n) =$ $\mathbf{\Theta(n^{\lg(3)}) \approx \Theta(n^{1.59})}$.

**c)** $T(n) = 4T(n/2) + n^2 sqrt(n) = 4T(n/2) + n^{5/2}$.
In the Master Theorem, $a = 4$, $b = 2$, $E = \lg(4) = 2$, $n^E = n^2$, and $f(n) = n^{5/2}$. $f(n) = \Omega(n^{E+\varepsilon})$, where we could take $\varepsilon = 0.1$. $af(n/b) = 4(n/2)^{5/2} = 2^{-1/2}n$, so $af(n/b) \le cf(n)$ where $c = 2^{-1/2} < 1$. The Master Theorem (case 3) tells us $T(n) = \Theta(f(n))$ = $\mathbf{\Theta(n^{5/2})}$.

**e)** $T(n) = 2T(n/2) + n/\lg(n)$.
In the notation of the Master Theorem, $a = 2$, $b = 2$, $E = \lg(2) = 1$, $f(n) = n/\lg(n)$, and $n^E = n$. The Master Theorem does not apply, as $f(n)$ grows too rapidly for case (1) and not rapidly enough for case (2). But the extension to the Master Theorem in the handout tells us that $T(n) =$ $\mathbf{\Theta(n \lg\lg(n))}$.

**h)** $T(n) = T(n-1) + \lg(n)$

$$
\begin{aligned}
T(n) &= T(n-1) + \lg(n) \\
&= T(n-2) + \lg(n-1) + \lg(n) \\
&= T(n-3) + \lg(n-2) + \lg(n-1) + \lg(n) \\
&\vdots \\
&= T(1) + \lg(2) + \lg(n-2) + \lg(n-1) + \lg(n) \\
&= T(1) + \lg(n!) \\
&= n\lg(n) - 1.44n + O(\lg(n)) = \mathbf{\Theta(n\lg(n))}
\end{aligned}
$$

**Exercise H.** Assume the recurrence

$$C(n) = C(0.8n) + C(0.5n) + C(0.2n) + n, \quad C(1) = 1$$

has a solution

$$C(n) = an^b + cn + d$$

for some real numbers $a$, $b$, $c$, and $d$. We ignore the fact that $0.2n$, $0.5n$, and $0.8n$ are not in general integers.Substituting the proposed solution into the recurrence, we obtain

$$
\begin{aligned}
an^b + cn + d \;=\; & a(0.2n)^b + c(0.2n) + d + \\
& a(0.5n)^b + c(0.5n) + d + \\
& a(0.8n)^b + c(0.8n) + d + n
\end{aligned}
$$

or

$$a(1 - 0.2^b - 0.5^b - 0.8^b)n^b + (c - 0.2c - 0.5c - 0.8c - 1)n - 2d = 0.$$

Since this must hold for all $n$, the coefficients of $n^b$, $n$, and 1 each must equal 0.

$$
\begin{aligned}
a(1 - 0.2^b - 0.5^b - 0.8^b) = 0 \;&\Rightarrow\; 1 - 0.2^b - 0.5^b - 0.8^b = 0 \quad (\text{since } a \ne 0), \\
c - 0.2c - 0.5c - 0.8c - 1 = 0 \;&\Rightarrow\; c = -2,
\end{aligned}
$$

$$2d = 0 \quad \Rightarrow \quad d = 0.$$

Now the condition C(1) = 1 implies $a + c + d = 1$, which together with the values of c and $d$ above gives $a = 3$.

Let $f(x) = 1 - 0.2^x - 0.5^x - 0.8^x$. $b$ is a root of $f(x)$. $f(x)$ is clearly an increasing function of $x$, since each of $0.2^x$, $0.5^x$, and $0.8^x$ decrease as $x$ increases. $f(1) = -0.5$ and $f(2) = 0.07$. Thus $f(x)$ has a unique root (equal to $b$) , which must lie between 1 and 2 (probably closer to 2), and it can be approximated by bisection or Newton's method. With an initial guess of 2, Newton's method with 4 iterations gives $b = 1.8267247$, or to two decimal places $\boldsymbol{b \approx 1.83}$.

*Note:* The complete solution is $C(n) = 3\,n^{1.83} - 2n$. We saw earlier that division into 3 equal-sized subproblems of total size $1.5n$, with linear divide/combine time, led to a $\Theta(n^{1.59})$ time algorithm. Here the division into three unequal-sized subproblems with the same total size raises the time to $\Theta(n^{1.83})$.

**Exercise I.** The inversions in the array **a** = (41, 16, 74, 33, 66, 54) are:

(41,16), (41,33), (74,33), (74,66), (74,54), (66,54).

The number of inversions is 6. Straight insertion sort would perform 6 comparisons in which it finds the elements out of order (1 for each inversion) and $n-1 = 5$ comparisons in which it finds the elements in order — a total of **11 comparisons**. Each comparison in which the elements are out of order is followed by an exchange, so there are **6 exchanges**.

**Exercise J**

**a)** **Not a strict weak order.** Here $(x,y) \sim (u,v)$ when

$$\textbf{not}(x < u \textbf{ and } y < v) \textbf{ and not}(u < x \textbf{ and } v < y).$$

Note $(0,0) \sim (2,-1)$ and $(2,-1) \sim (1,2)$, but $(0,0) \not\sim (1,2)$, so $\sim$ is not transitive.

**b)** **A strict weak order.** The equivalence classes consist are the ellipses centered at the origin, with semi-major axis along the X-axis, and semi-minor axis equal to half the semi-major axis.

**c)** **A strict weak order.** $(x,y) \sim (u,v)$ when $x - y = u - v$. For each real number $\alpha$, there is an equivalence class consisting of those points $(x,y)$ for which $x - y = \alpha$, or equivalently, $y = x - \alpha$. This class is the line with slope 1 and Y-intercept $-\alpha$.

**d)** **Not a strict weak order.** Note $(x,y) \sim (u,v)$ when **not**$(x < u - 1)$ **and not**$(u < x - 1)$. Thus $(x,y) \sim (u,v)$ when $|x - u| \le 1$. Note $(0,0) \sim (1,0)$ and $(1,0) \sim (2,0)$, but $(0,0) \not\sim (2,0)$, so $\sim$ is not transitive.

**e)** **A strict weak order.** For each integer $i$, there is an equivalence class consisting of $\{ r \mid r \text{ is a real number with } i \le r < i+1 \}$.

**f)** **A strict weak order.** For each real number $\alpha$ in [0,1), there is an equivalence class consisting of $\{\alpha + i \mid i \text{ is an integer}\}$.

**g)** **Not a strict weak order.** Note $a \sim b$ if neither $a$ nor $b$ is a proper divisor of the other. So $2 \sim 3$ and $3 \sim 4$, but $2 \not\sim 4$. This means $\sim$ is not transitive.