An Extension to The Master Theorem

In the Master Theorem, as given in the textbook and previous handout, there is a gap between cases (1) and (2), and a gap between cases (2) and (3).

For example, if $a = b = 2$ and $f(n) = n/\lg(n)$ or $f(n) = n\lg(n)$, none of the cases apply. The extension below partially fills these gaps.

THEOREM (*Extension of Master Theorem*) If a, b, $E \overset{\text{def}}{=} \log_b(a)$, and $f(n)$ are as in the Master Theorem, the recurrence

$$T(n) = a\,T(n/b) + f(n),\ \ T(1) = d,$$

has solution as follows:

1') If $f(n) = O(n^E(\log_b n)^\alpha)$ with $\alpha < -1$, then $T(n) = \Theta(n^E)$.

2') If $f(n) = \Theta(n^E(\log_b n)^{-1})$, then $\mathrm{T}(n) = \Theta\big(n^E \log_b \log_b(n)\big)$.

3') If $f(n) = \Theta\big(n^E(\log_b n)^\alpha\big)$ with $\alpha > -1$, then $\mathrm{T}(n) = \Theta\big(n^E(\log_b n)^{\alpha+1}\big)$.

4') [*same as in Master Theorem*] If $f(n) = \Omega(n^{E+\varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(f(n))$, provided there is a constant c with $c < 1$ such that

$$a\,f(n/b) \le c\,f(n)\ \text{ for all } n \text{ sufficiently large.}$$

Note: (1') above includes case (1) of the Master Theorem.

(3') above with $\alpha = 0$ is case (2) in the Master Theorem.

We make use of the fact below, which follows from the close connection between sums and integrals.

LEMMA. $\sum_{i=1}^{\infty} i^{\alpha}$ converges if $\alpha < -1$ and diverges otherwise. $\sum_{i=1}^{n} i^{\alpha} \approx \ln(n)+\gamma$ if $\alpha = -1$, and $\sum_{i=1}^{n} i^{\alpha} \approx n^{\alpha+1}/(\alpha+1)$ if $\alpha > -1$.

Proof of the extended Master Theorem when *n* is a power of b.

Case (4) is exactly as in the Master Theorem, so we consider only (1), (2), and (3). In case 1, $f(n) \le \Theta(n^E(\log_b n)^{\alpha})$. In cases (2) and (3), $f(n) = \Theta(n^E(\log_b n)^{\alpha})$ for some α.

Let $n = b^k$, so $k = \log_b(n)$. From the previous handout, we know that

$$T(n) = f(n) + af(n/b) + a^2 f(n/b^2) + \ldots + a^{k-1} f(n/b^{k-1}) + a^k d.$$

Putting $f(n) \approx cn^E(\log_b n)^{\alpha}$ for some constant c, we get

$$\begin{aligned} T(n) \approx\ & cn^E(\log_b n)^{\alpha} \\ & + ac(n/b)^E(\log_b(n/b))^{\alpha} \\ & + a^2 c(n/b^2)^E(\log_b(n/b^2))^{\alpha} \\ & + \ldots \\ & + a^{k-1} c(n/b^{k-1})^E(\log_b(n/b^{k-1}))^{\alpha} \\ & + a^k d \end{aligned}$$

(In case 1, this is just an upper bound for $T(n)$.)

Note $a^k = n^E$. Also $n = b^k$, so $\log_b(n/b^i) = \log_b(b^{k-i}) = k-i$. Finally, note $a = b^E$, so in $a^i c(n/b^i)^E$ in the formula above, $a^i / b^{iE} = 1$.

With these simplifications, our formula becomes

$$T(n) \approx cn^E k^\alpha + cn^E(k-1)^\alpha + cn^E(k-2)^\alpha + \ldots + cn^E 1^\alpha + dn^E$$
$$= cn^E \sum_{i=1}^{k} i^\alpha + dn^E$$

If $\alpha < -1$, then $1 \le \sum_{i=1}^{k} i^\alpha < c'$, where $c' = \sum_{i=1}^{\infty} i^\alpha =$ some constant. So at worst $T(n) \approx (cc'+d)n^E = \Theta(n^E)$. But in the handout on the Master Theorem we remarked that $T(n)$ can never be less than $\Theta(n^E)$, since the bottom level alone requires this much time.

If $\alpha = -1$, then $\sum_{i=1}^{k} i^\alpha \approx \ln(k) = \ln \log_b(n) = q \log_b\log_b(n)$ for some constant q, so

$$T(n) \approx cq\, n^E \log_b\log_b(n) + dn^E = \Theta(n^E \log_b\log_b(n)).$$

If $\alpha > -1$, then $\alpha+1 > 0$, and $\sum_{i=1}^{k} i^\alpha \approx k^{\alpha+1}/(\alpha+1) = \log_b(n)^{\alpha+1}/(\alpha+1)$, so

$$T(n) \approx c\,n^E \log_b(n)^{\alpha+1}/(\alpha+1) + dn^E = \Theta(n^E (\log_b(n))^{\alpha+1}).$$