Factorials

We define $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$ if $n$ is a nonnegative integer.

An empty product is normally defined to be 1.

With this convention, $0! = 1$.

An alternative is to define $n!$ recursively on the nonnegative integers.

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n(n-1)! & \text{if } n \geq 1. \end{cases}$$

As $n$ increases, $n!$ increases very rapidly (exponentially).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>120</td>
</tr>
<tr>
<td>10</td>
<td>3628800</td>
</tr>
<tr>
<td>15</td>
<td>$1.307674 \times 10^{12}$</td>
</tr>
<tr>
<td>20</td>
<td>$2.432902 \times 10^{18}$</td>
</tr>
<tr>
<td>30</td>
<td>$2.652529 \times 10^{32}$</td>
</tr>
<tr>
<td>40</td>
<td>$8.159153 \times 10^{47}$</td>
</tr>
<tr>
<td>50</td>
<td>$3.041409 \times 10^{64}$</td>
</tr>
<tr>
<td>60</td>
<td>$8.320987 \times 10^{81}$</td>
</tr>
<tr>
<td>70</td>
<td>$1.197857 \times 10^{100}$</td>
</tr>
<tr>
<td>80</td>
<td>$7.156946 \times 10^{118}$</td>
</tr>
</tbody>
</table>

For any fixed number $a$, $n! > a^n$ for all $n$ sufficiently large.

On the other hand, $n! < n^n$ for all $n$.

Stirling’s Formula provides a good approximation to $n!$ in closed form:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

If $S_0(n)$ denotes $\left(\frac{n}{e}\right)^n \sqrt{2\pi n}$, then $\lim_{n \to \infty} S_0(n) / n! = 1$.

In fact, the limit approaches 1 quite rapidly as $n$ increases.

When $n = 5$, $S_0(n) / n! = 0.9835$.
When $n = 10$, $S_0(n) / n! = 0.9917$.
When $n = 50$, $S_0(n) / n! = 0.9983$.

An even better approximation is obtained by multiplying $S_0(n)$ by $1 + 1/(12n)$.

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left(1 + \frac{1}{12n}\right)$$

If $S_1(n)$ denotes $\left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left(1 + \frac{1}{12n}\right)$, then

When $n = 1$, $S_1(n) / n! = 0.998982$.
When $n = 5$, $S_1(n) / n! = 0.999883$.
When $n = 10$, $S_1(n) / n! = 0.999968$.
When $n = 50$, $S_1(n) / n! = 0.999999$.

Here are the approximations to $n!$ for the values of $n$ in the previous table.
Previously, we mentioned that \( n! \) grows more rapidly than \( a^n \) (\( a \) fixed) but less rapidly than \( n^n \).

By Stirling’s formula, \( n! \) grows about as rapidly as \((n/e)^n\).

Stirling’s formula also gives a good approximation to \( \lg(n!) \):

\[
\lg(n!) \approx n \lg(n) - n \lg(e) + 0.5 \lg(2\pi) + \lg(e)/(12n)
\]

or

\[
\lg(n!) \approx n \lg(n) - 1.44n
\]

We sometimes write \( \lg(n!) \approx n \lg(n) \), but the 1.44\( n \) term never becomes negligible for practical values of \( n \).

### Why is \( n! \) important in algorithms?

\( n! \) is the number of permutations of an \( n \)-element sequence with distinct elements. In other words, it is the number of ways to arrange \( n \) distinct objects.

For example, there are \( 4! = 24 \) ways to arrange the letters a, b, c, d:

\[
\begin{align*}
&abcd \quad bacd \quad cbad \quad dabc \\
&abdc \quad bdac \quad cdab \quad dcab \\
&acbd \quad bcad \quad cbad \quad dbac \\
&acdb \quad bcda \quad cdab \quad dbca \\
&adbc \quad bdac \quad cdab \quad dcab \\
&adcb \quad bdca \quad cdab \quad dcba \\
\end{align*}
\]

Any algorithm that looks at every possible arrangement of \( n \) objects would take time at least proportional to \( n! \) (and thus be practical only for very small \( n \)— say \( n \) less than 15 or 20).

What if we have \( n \) elements that are not distinct? Say there are \( k \) distinct elements, occurring with frequencies \( n_1, n_2, \ldots, n_k \), where \( n_1 + n_2 + \ldots + n_k = n \). The number of arrangements is

\[
\frac{n!}{n_1! n_2! \ldots n_k!}
\]

Thus there are \( 5! / (3! 1! 1!) = 20 \) ways to arrange a, a, a, b, c:

\[
\begin{align*}
aaabc & \quad aacab & \quad abca & \quad baaac & \quad caaab \\
aaacb & \quad aacba & \quad abca & \quad baaca & \quad caaba \\
ababc & \quad abaac & \quad abaca & \quad baca & \quad caba \\
abaca & \quad abaca & \quad acba & \quad baca & \quad caaa \\
\end{align*}
\]
We have defined $n!$ only on the nonnegative integers, but we can extend to the nonnegative real numbers (as well as certain negative real numbers).

Consider \[
\int_0^\infty t^x e^{-t} \, dt, \text{ where } x \text{ is any nonnegative real number.}
\]
(Actually, we only need $x > -1$.)

The value of the integral depends on $x$, so denote it by $g(x)$.

\[
\begin{align*}
g(0) &= \int_0^\infty t^0 e^{-t} \, dt = -e^{-t} \bigg|_0^\infty = -0 - (-1) = 1 \\
\int_0^\infty tx e^{-t} \, dt &= \int_0^\infty u(t) v'(t) \, dt \\
&= u(\infty)v(\infty) - u(0)v(0) - \int_0^\infty u'(t)v(t) \, dt \\
&= 0 - 0 - x \int_0^\infty t^{x-1}(-e^{-t}) \, dt \\
&= x \int_0^\infty t^{x-1} e^{-t} \, dt = xg(x-1).
\end{align*}
\]

Now $g(0) = 1$ and $g(x) = xg(x-1)$ for all $x > 0$ implies $g(x) = x!$ whenever $x$ is a nonnegative integer. So it is natural to define

$$x! = \int_0^\infty t^x e^{-t} \, dt$$

for all nonnegative real numbers $x$.

Actually, this definition makes sense for $x > -1$. When $x = -1$, the integral diverges.

One can show that

\[
\begin{align*}
(1/2)! &= \sqrt{\pi}/2 \approx 0.8862 \\
(3/2)! &= (3/2)(1/2)! = 3\sqrt{\pi}/4 \approx 1.3293 \\
(5/2)! &= (5/2)(3/2)! = 15\sqrt{\pi}/8 \approx 3.3234 \\
(-1/2)! &= (1/2)! / (1/2) = \sqrt{\pi} = 1.7724
\end{align*}
\]

Note: The function we defined as $g(x)$ is essentially the Gamma function $\Gamma(x)$, introduced by Euler.

However, $\Gamma(x)$ is defined as $\int_0^\infty t^{x-1} e^{-t} \, dt$ whenever $x > 0$.

So $x! = \Gamma(x+1)$ whenever $x > -1$. 
