**Summation by Parts**

An important technique of calculus is integration by parts:

\[
\int_a^b u(x)v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx
\]

This is useful, obviously, when \( u'(x)v(x) \) is easier to integrate than \( u(x)v'(x) \), e.g., if \( u(x) = x \) and \( v(x) = e^x \).

An analogous technique, called summation by parts, works for sums. One version of the summation by parts formula is:

\[
\sum_{i=a+1}^b u_i (v_i - v_{i-1}) = u_b v_b - u_a v_a - \sum_{i=a+1}^b (u_i - u_{i-1}) v_{i-1}
\]

**Example:** Find \( \sum_{i=1}^n i 2^i \).

Set \( a = 0 \), \( b = n \), \( u_i = i \), and \( v_i = 2^{i+1} \).

Then \( u_i - u_{i-1} = 1 \), \( v_i - v_{i-1} = 2^{i+1} - 2^i = 2^i \), and

\[
\sum_{i=1}^n i 2^i = \sum_{i=a+1}^b u_i (v_i - v_{i-1})
= u_b v_b - u_a v_a - \sum_{i=a+1}^b (u_i - u_{i-1}) v_{i-1}
= n 2^{n+1} - 0 \cdot 2^1 - \sum_{i=1}^n 1 \cdot 2^i
= n 2^{n+1} - (2^{n+1} - 2)
= (n - 1) 2^{n+1} + 2
\]